Undercut-Proof Equilibria*

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June 19, 2000

Abstract

We propose an equilibrium concept, called Undercut-Proof equilibrium, for price competition between firms producing differentiated brands. We demonstrate in this environment that, whereas a Nash-Bertrand equilibrium in pure actions never exists, a unique Undercut-Proof equilibrium always exists and has the following properties: (a) Brands' prices monotonically diverge when the brands become more differentiated, and are identical when the brands become homogeneous. (b) The firm with the larger market share charges a lower price than the firm with the smaller market share but earns a larger profit. The Undercut-proof equilibrium is easily calculable, and supports an upper bound on colluding prices in a dynamic meeting-the-competition price game.

Keywords: Undercut-Proof Equilibrium, Differentiated Products, Conjectural Variations

JEL Classification Numbers: L1, D43

(Draft = conhot71.tex 2000/06/19 20:40)

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*We wish to thank seminar participants at Northwestern University, Stockholm School of Economics, University of Texas at Austin, and the participants in the "Strohs-Labatts" conference of the Michigan and Western Ontario Economics Departments, held in London, Ontario for most valuable comments and suggestions. The work on this paper has began while Oz Shy was visiting the Department of Economics at the University of Michigan. We thank Motty Perry, Sougata Poddar, David Roth, and Jörgen Weibull for most helpful discussions.

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1. Introduction

The goal of this paper is to explore the simplest possible differentiated products environment where (pure) Nash-Bertrand equilibrium prices do not exist due to price cycles a la Edgeworth and to suggest an alternative equilibrium concept as better suited to analyzing such environments.

We develop and characterize a concept called an *Undercut-Proof equilibrium*. In an Undercut-Proof equilibrium, each firm chooses its price so as to maximize profit while ensuring that its price is sufficiently low that any rival firm would *not* find it profitable to set a lower price in order to grab all of the first firm's customers. Thus, unlike the Nash-Bertrand behavior, where each firm assumes that the rival firm does not alter its price, in an Undercut-Proof equilibrium environment, firms assume that rival firms are more sophisticated in that they are 'ready' to reduce their prices whenever undercutting and grabbing their rivals' customers is profitable. We believe that these beliefs are pervasive amongst firms competing in differentiated products using pricing strategies.

For the environment we consider we show that a unique Undercut-Proof equilibrium always exists. The equilibrium prices of brands monotonically diverge when the brands become more differentiated and are identical when the brands are homogeneous. Also, firms with larger market shares charge lower prices but earn higher profits than do firms with smaller market shares. This is because firms with larger market shares are better targets to be undercut by smaller firms. Note that while these properties are observed in any retail industry, they are not predictions of a wide variety of models, including the basic Hotelling (1929) linear city model. Finally, the Undercut-Proof equilibrium can be calculated easily for any number of firms in the industry.

Our paper develops the Undercut-Proof equilibrium concept for a wide variety of simple environments and applications in which Nash-Bertrand equilibria in prices do not exist. The simple example analyzed in Section 2 can be applied easily to a wide variety of problems. First, consider an incumbent airline providing a nonstop direct service from city A to city B.

Now consider a potential entrant who can offer at a lower price an indirect service from city A to B via a hub located in city C. Clearly, travelers with low values of time will fly with the entrant, whereas travelers with high values of time will be willing to pay a higher price and fly nonstop. It turns out that this simple model does not have a Nash equilibrium in prices after entry occurs. In fact, as is recognized by most authors in the entry-deterrence literature, pure Nash equilibria in prices generally do not exist when the entrant introduces a product which is differentiated from that offered by the incumbent, especially in cases where there is only a finite number of types of consumers. For this reason, authors of entry models focus their analyses on Cournot competition after entry occurs. In our opinion, for some industries such as transportation and communication, price competition after entry occurs is more appropriate, since "hit-and-run" type of entry is commonly observed in these types of industries.

Second, consider a discrete location model where a city is located on both banks of a river. On each bank there is one store. Some residents live on each bank and there is a bridge with a fixed toll that must be paid in order to cross the river. The example analyzed in Section 2 shows that there does not exist a pure Nash-Bertrand equilibrium for this least sophisticated location/product differentiation model; however, a unique Undercut-Proof equilibrium does exist.

Third, consider an environment where consumers are "locked-in" to a certain brand after the first purchase. For example, once consumers buy a computer brand, they invest a substantial amount of time which makes it beneficial for them to buy the same brand on their second purchase. Given these switching costs, Klemperer (1987) showed that a pure Nash-Bertrand equilibrium in prices of two brand-producing firms could exist only for a sufficiently large switching cost. However, the Undercut-Proof equilibrium exists for any value of the switching cost.

The nonexistence of a pure-action Bertrand-Nash equilibrium due to the emergence of price cycles was first identified by Edgeworth for the case of homogeneous products where firms have capacity constraints. It turns out that simple differentiated products models “suffer” from the same cycles even without capacity constraints. Edgeworth Cycles describe price competition where, at ‘high’ prices, each firm can increase its profit by undercutting the price set by its rivals, but at ‘low’ prices, each firm can increase its profit by raising its price. In Edgeworth’s words (1897):

> “In the last case there will be an intermediate tract through which the index of value will oscillate, or rather vibrate irregularly for an indefinite length of time. There will never be reached that determinate position of equilibrium which is characteristic of perfect competition.”

Edgeworth discovered cyclical price movements in his attempt to resolve the so-called Bertrand Paradox by introducing capacity constraints into the Bertrand price game. While Edgeworth cycles are generally associated with price competition under capacity constraints (see discussions in Tirole [1988], pp. 211, 233–234), these price cycles occur also in differentiated products models without capacity constraints.

Finally, one may ask why we believe that this equilibrium concept is needed, rather than using some solution for dynamic games. First, note that a repeated game can have too many equilibria depending on different penalty functions. Second, another disadvantage of using a repeated-game solution is that it is too complicated for undergraduate students and applied-research economists. Thirdly, even if we simplify the dynamic game and use some kind of an alternating-moves Markov perfect equilibrium (see Eaton and Engers, 1990, for example) alternating moves do not have much sense in multi-firm price competition since the order of moves becomes ad-hoc. Another problem, is that a MPE generally requires an assumption about a reservation price, whereas the proposed equilibrium always exists even without assuming reservation prices. Lastly, we demonstrate some nice properties of this equilibrium which do provide some good prediction on price competition between stores or firms selling differentiated products.

The paper is organized as follows: Section 2 constructs a simple example of a differentiated products market and demonstrates that an Undercut-Proof equilibrium always exists and has certain properties commonly observed in markets for differentiated products. Section 3 generalizes the model to any number of firms, defines the Undercut-Proof equilibrium, proves existence and uniqueness, derives an algorithm for calculating this equilibrium, and characterizes the equilibrium properties. Section 4 illustrates the equilibrium and the characterization algorithm in several interesting n-firm examples. Section 5 provides a dynamic justification for the Undercut-Proof equilibrium by demonstrating how the UPE prices constitute upper bounds on colluding prices under a dynamic 'meeting-the-competition' game; and also as upper bounds on resale price maintenance price ceilings imposed by a manufacturer selling to two dealers separated by some distance. Section 6 concludes with a discussion and interpretations of the Undercut-Proof equilibrium prices.

2. Why is the Undercut-Proof Equilibrium Needed?

Consider the following example (see Shilony 1977, Eaton and Engers 1990, and Shy 1996, Ch. 7), of a market with two stores called A and B which sell differentiated brands. Assume that production costs are zero. There are two groups of consumers, type α (called brand A oriented consumers) and type β (called brand B oriented consumers). There are $N_\alpha > 0$ type α consumers and $N_\beta > 0$ type β consumers.

Each consumer buys one unit either from store A or store B. Let p_A and p_B denote the prices of the stores and let $T \geq 0$ denote the extra distaste cost a consumer bears if he buys his less preferred brand. Altogether, the utilities of consumers of type α and type β are assumed to be

$$U_\alpha \stackrel{\text{def}}{=} \begin{cases} -p_A & \text{buying from } A \\ -p_B - T & \text{buying from } B \end{cases} \quad \text{and} \quad U_\beta \stackrel{\text{def}}{=} \begin{cases} -p_A - T & \text{buying from } A \\ -p_B & \text{buying from } B. \end{cases} \tag{1}$$

One way of interpreting this example is as a discrete version of the Hotelling (1929) location model where the two stores locate on opposite sides of a lake or high terrain and where crossing from one side to the other requires paying a fixed transportation cost of T.

Let n_A denote the (endogenously determined) number of consumers buying from store A, and n_B denote the number of consumers buying from store B. Then, (1) implies that

$$n_A = \begin{cases} 0 & \text{if } p_A > p_B + T \\ N_\alpha & \text{if } p_B - T \leq p_A \leq p_B + T \\ N_\alpha + N_\beta & \text{if } p_A < p_B - T \end{cases} \quad \text{and } n_B = \begin{cases} 0 & \text{if } p_B > p_A + T \\ N_\beta & \text{if } p_A - T \leq p_B \leq p_A + T \\ N_\alpha + N_\beta & \text{if } p_B < p_A - T. \end{cases} \tag{2}$$

2.1 Nonexistence of a Nash-Bertrand equilibrium

A Nash-Bertrand equilibrium is the nonnegative pair $\left(p_A^N, p_B^N\right)$ such that, for a given p_B^N, store A chooses p_A^N to maximize $\pi_A \stackrel{\text{def}}{=} p_A n_A$ and, for a given p_A^N, store B chooses p_B^N to maximize $\pi_B \stackrel{\text{def}}{=} p_B n_B$, where n_A and n_B are given in (2).

Proposition 1

There does not exist a Nash-Bertrand equilibrium in pure price-strategies for the differentiated products model.[1]

Proof. To establish a contradiction, suppose that $\left(p_A^N, p_B^N\right)$ is a Nash equilibrium. Then, there are three cases: (i) $|p_A^N - p_B^N| > T$, (ii) $|p_A^N - p_B^N| < T$ and (iii) $|p_A^N - p_B^N| = T$.

(i) With no loss of generality, suppose that $p_A^N - p_B^N > T$. Then (2) implies that $n_A^N = 0$, and hence $\pi_A^N = 0$. However, store A can increase its profit by reducing its price to $\tilde{p}_A = p_B^N + T$, in which case $\tilde{n}_A = N_\alpha$ and $\tilde{\pi}_A = N_\alpha(p_B^N + T) > 0$; a contradiction.

(ii) With no loss of generality, suppose that $p_A^N < p_B^N + T$. Then store A can increase its profit by slightly increasing its price to $\tilde{p}_A$ satisfying $p_A^N < \tilde{p}_A < p_B^N + T$ to earn a profit level of $\tilde{\pi}_A = N_\alpha \tilde{p}_A > \pi_A^N$; a contradiction.

(iii) With no loss of generality, suppose that $p_A^N - p_B^N = T$. Then, $p_B^N = p_A^N - T < p_A^N + T$ and store B can increase its profit by slightly raising p_B^N; a contradiction. ■

2.2 The Modified Zero Conjectural Variations (ZCV) equilibrium

The Modified Zero Conjectural Variations (ZCV) equilibrium was developed to address the nonexistence of an equilibrium in the continuous Hotelling (1929) model for the case where

[1] Shilony (1977) proves existence of a Nash equilibrium with mixed actions for consumers with a reservation utility. As it turns out, the characterization of the equilibrium is very complicated.

the firms are located 'close' to each other. The ZCV equilibrium is a Nash-Bertrand equilibrium restricted to action sets that exclude undercutting; see Eaton and Lipsey (1978) and Novshek (1980). More precisely, the idea amounts to restricting the set of actions to prices yielding nonzero market shares. Following Gabszewicz and Thisse (1986; p. 31, footnote 31), we now define a ZCV equilibrium for the present environment.

Definition 1

The pair of prices $\left(p_A^Z, p_B^Z\right)$ *is called a* ***Zero Conjectural Variation equilibrium (ZCV)*** *if it is a Nash equilibrium on the set of prices in which each firm has a strictly positive market share. Formally,*

$$p_A^Z n_A(p_A^Z, p_B^Z) \geq p_A n_A(p_A, p_B^Z) \quad \text{and} \quad p_B^Z n_B(p_A^Z, p_B^Z) \geq p_B n_B(p_A^Z, p_B)$$

$$\text{for all} \quad (p_A, p_B) \in \{(p_A, p_B) \mid n_A(p_A, p_B) > 0,\ n_B(p_A, p_B) > 0\}.$$

Note that according to this definition the action set of one player depends on the action set of another. The following proposition shows that this restriction of the action sets does not solve the existence problem since in this game stores maximize profit by setting price equals to the rival store's price plus the T, which generate an inconsistency with equilibrium.

Proposition 2

There does not exist a ZCV equilibrium

Proof. Since undercutting is ruled out by Definition 1, for a given p_B^Z, firm A maximizes profit by setting $p_A^Z = p_B^Z + T$ Similarly, for a given p_A^Z, firm B maximizes profit by setting $p_B^Z = p_A^Z + T$. Clearly both equations cannot be satisfied simultaneously. ■

It should be pointed out, that a ZCV could exist if consumers have reservation utilities. In such a case, the ZCV-equilibrium prices would be equal the reservation utilities.

2.3 The Undercut-Proof equilibrium: An illustration

Without providing a formal definition, we now *illustrate* how an Undercut-Proof equilibrium can be *calculated*. Consider the following behavior of two firms:

1. For given p_B^U and n_B^U, firm A chooses the highest price p_A^U subject to

$$\pi_B^U = p_B^U n_B^U \geq (p_A - T)(N_\alpha + N_\beta).$$

2. For given p_A^U and n_A^U, firm B chooses the highest price p_B^U subject to

$$\pi_A^U = p_A^U n_A^U \geq (p_B - T)(N_\alpha + N_\beta).$$

3. The distribution of consumers between the firms is determined in (2).

The first part states that, in an Undercut-Proof equilibrium, firm A sets the highest price it can while preventing firm B from undercutting p_A^U and grabbing firm A's customers. More precisely, firm A sets p_A^U as high as possible without causing B's equilibrium profit level to be smaller than B's profit level when it undercuts by setting $\tilde{p}_B = p_A^U - T$, and selling to $\tilde{n}_B = N_\alpha + N_\beta$ customers. The above two inequalities therefore hold as equalities which can be solved for the equilibrium prices

$$p_A^U = \frac{(N_\alpha + N_\beta)(N_\alpha + 2N_\beta)T}{(N_\alpha)^2 + N_\alpha N_\beta + (N_\beta)^2} > T \text{ and } p_B^U = \frac{(N_\alpha + N_\beta)(2N_\alpha + N_\beta)T}{(N_\alpha)^2 + N_\alpha N_\beta + (N_\beta)^2} > T. \qquad (3)$$

First note that by setting $p_i \leq T$, each firm can secure a strictly positive market share without being undercut. Hence, in an Undercut-Proof equilibrium both firms maintain a strictly positive market share. Substituting (3) into (2), we have that $n_A^U = N_\alpha$ and $n_B^U = N_\beta$.

Figure 1 illustrates how the Undercut-Proof equilibrium is determined. Figure 1's left panel, shows how firm A is constrained in setting p_A so that firm B cannot benefit from undercutting p_A^U. Figure 1's middle panel shows how firm B is constrained in setting p_B so that firm A would not benefit from undercutting p_B^U. Figure 1's right panel displays the region where neither firm finds it profitable to undercut its rival; the Undercut-Proof equilibrium prices maximize profits on this region. It should be emphasized that the curves drawn in Figure 1 are *not* best response (reaction) functions but simply divide the regions into prices that make undercutting profitable or unprofitable for each firm.

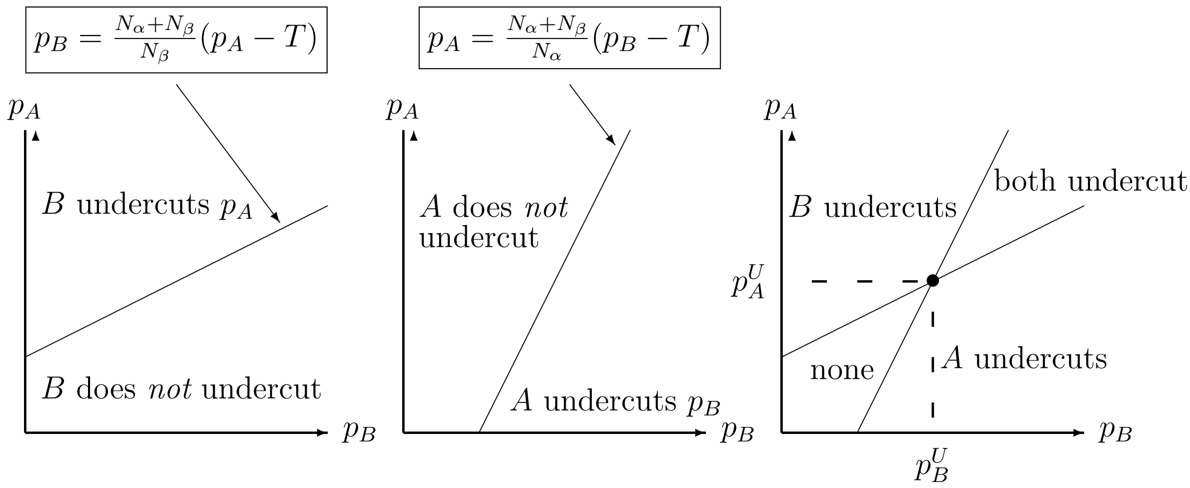

Figure 1: Undercut-Proof equilibrium

2.4 Four important properties of the Undercut-Proof equilibrium

We now conclude this example with characterizations of the Undercut-Proof equilibrium prices. First, from (3), prices rise with distaste (transportation) costs and monotonically decline to zero as distaste costs approach zero, reflecting a situation in which the products become homogeneous.

Second,

$$\Delta p^U \stackrel{\text{def}}{=} p_B^U - p_A^U = \frac{[(N_\alpha)^2 - (N_\beta)^2]\,T}{(N_\alpha)^2 + N_\alpha N_\beta + (N_\beta)^2} < T. \tag{4}$$

Hence, $\Delta p^U \geq 0$ if and only if $N_\alpha \geq N_\beta$. Thus, in an Undercut-Proof equilibrium, the store selling to the larger number of consumers charges a lower price. This result is commonly observed in retailing, where discount stores sell to larger numbers of consumers (e.g., WalMart and Kmart). Note that this result is *not* obtained in the conventional Hotelling linear-city location model which predicts that the store with the higher market share sells at a higher price.

Third,

$$\Delta \pi^U \stackrel{\text{def}}{=} \pi_B^U - \pi_A^U = p_B^U N_\beta - p_A^U N_\alpha = \frac{(N_\alpha + N_\beta)^2 (N_\beta - N_\alpha) T}{(N_\alpha)^2 + N_\alpha N_\beta + (N_\beta)^2}. \tag{5}$$

Hence, $\Delta \pi^U \geq 0$ if and only if $N_\beta \geq N_\alpha$. That is, in an Undercut-Proof equilibrium, the firm selling to a larger number of consumers makes a higher profit despite selling at a lower price.

Fourth, under a symmetric distribution of consumers ($N_\alpha = N_\beta$), the equilibrium prices are given by $p_A^U = p_B^U = 2T$. That is, each firm can mark up its price to twice the level of the distaste (transportation) cost without being undercut.

3. The General Model

The market is serviced by a differentiated products industry with ϕ firms indexed by $i \in \{1, 2, \ldots, \phi\}$. There are N_i brand i oriented consumers with a utility function

$$U_i \stackrel{\text{def}}{=} \begin{cases} -p_i & \text{if buying brand } i \\ -p_j - T & \text{if buying brand } j, \text{ for all } j \neq i. \end{cases} \tag{6}$$

With no loss of generality, assume that brand 1 is the least popular brand, brand 2 is the second least popular brand, and so on with brand ϕ being the most popular. Formally, we let

$$0 < N_1 \leq N_2 \leq \cdots \leq N_\phi. \tag{7}$$

Let $n_i(p_1, \ldots, p_\phi)$ denote the number of consumers purchasing brand i under this vector of prices; $i = 1, \ldots, \phi$. In order to determine $n_i(\cdot)$, for each firm i define

$$U(i,p) \stackrel{\text{def}}{=} \left\{ j \in \{1, \ldots, \phi\} \mid p_j < p_i - T,\ p_j = \min_{k=1,\ldots,\phi} p_k, \right\}.$$

Let $\#U(i,p)$ denote the number of elements in $U(i,p)$. $U(i,p)$ is the set containing the indices of the firms which 'most severely' undercut firm i and $\#U(i,p)$ is the number of such firms. Also define

$$D(i,p) \stackrel{\text{def}}{=} \{ j \in \{1, \ldots, \phi\} \setminus \{i\} \mid i \in U(j,p) \}$$

which is the set of all the firms which are most severely undercut by firm i.

If firm i is undercut by any firm j then the number of customers serviced by firm i is zero. This is so even if firm i undercuts some other firm k since then firm j undercuts firm k even more severely than does firm i and gathers the customers from both firms i and k. If more than one firm equally most severely undercuts firm i (*i.e.* $\#U(i,p) \geq 2$) then the presumption is that the N_i customers of type-i are allocated equally to the $\#U(i,p)$ firms with indices in $U(i,p)$. The number of buyers serviced by firm i is therefore

$$n_i(p) = \begin{cases} 0 & \text{if } U(i,p) \neq \emptyset \\ N_i + \displaystyle\sum_{j \in D(i,p)} \frac{N_j}{\#U(j,p)} & \text{if } U(i,p) = \emptyset \end{cases} \qquad i = 1, \ldots, \phi. \tag{8}$$

The action set for each firm i is $\mathbb{R}_+$. An action $p_i \in \mathbb{R}_+$ is a choice by firm i of a price for its product. All prices are chosen simultaneously in the full-information, static game played by the firms.

3.1 Defining the Undercut-Proof equilibrium

In this subsection we define and motivate the Undercut-Proof equilibrium.

Definition 2

An ***Undercut-Proof equilibrium*** (UPE) *is a price vector* $p^U = (p_1^U, p_2^U, \ldots, p_\phi^U)$ *such that,*

I. Undercut Prevention: *For each i, given p_{-i}^U, $p_i = p_i^U$ maximizes p_i subject to*

$$p_j^U N_j \geq (p_i - T)(N_j + N_i) \quad \textit{for every firm } j,\ j \neq i; \textit{ and}$$

II. Consumer Equilibrium: *Given p^U, N_i $(i = 1, \ldots, \phi)$ solve (8).*

The first constraint states that firm i sets its price as high as possible without making it profitable for firm j to lower p_j to $p_i - T$ and grab the type-i customers. Note that in this multi-firm environment each firm perceives its ability to undercut only one firm at a time (and not a group of firms). Although it is possible that some firms/stores will be engaging in a 'grand undercutting' we believe that most stores are engaged in a single-store undercutting. For example, in a "meeting-the-competition" game, store reduces its price when informed of one store selling below its price.

3.2 Existence and uniqueness of the Undercut-Proof Equilibrium

We now proceed with proving existence and uniqueness of the Undercut-Proof equilibrium. The system of constraints given in Definition 2 can be written as

$$\begin{aligned}
p_1 &= T + \min\left\{\frac{N_2 p_2}{N_1+N_2}, \frac{N_3 p_3}{N_1+N_3}, \cdots, \frac{N_\phi p_\phi}{N_1+N_\phi}\right\} \\
p_2 &= T + \min\left\{\frac{N_1 p_1}{N_2+N_1}, \frac{N_3 p_3}{N_2+N_3}, \cdots, \frac{N_\phi p_\phi}{N_2+N_\phi}\right\} \\
\vdots \quad &\vdots \qquad\qquad \vdots \\
p_\phi &= T + \min\left\{\frac{N_1 p_1}{N_\phi+N_1}, \frac{N_2 p_2}{N_\phi+N_2}, \cdots, \frac{N_{\phi-1} p_{\phi-1}}{N_\phi+N_{\phi-1}}\right\}.
\end{aligned} \tag{9}$$

The system of constraints (9) defines the mapping $\Theta : \mathbb{R}_+^\phi \mapsto \mathbb{R}_+^\phi$ where

$$\Theta\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_\phi \end{pmatrix} = \begin{pmatrix} T + \min\left\{\frac{N_2 p_2}{N_1+N_2}, \frac{N_3 p_3}{N_1+N_3}, \cdots, \frac{N_\phi p_\phi}{N_1+N_\phi}\right\} \\ T + \min\left\{\frac{N_1 p_1}{N_2+N_1}, \frac{N_3 p_3}{N_2+N_3}, \cdots, \frac{N_\phi p_\phi}{N_2+N_\phi}\right\} \\ \vdots \\ T + \min\left\{\frac{N_1 p_1}{N_\phi+N_1}, \frac{N_2 p_2}{N_\phi+N_2}, \cdots, \frac{N_{\phi-1} p_{\phi-1}}{N_\phi+N_{\phi-1}}\right\} \end{pmatrix}. \tag{10}$$

We now prove a crucial proposition.

Proposition 3

Θ is a contraction mapping.

Proof. The distance between any two points $p', p'' \in \mathbb{R}_+^\phi$ is

$$d(p', p'') = \max_{1 \le i \le \phi} \|p'_i - p''_i\|.$$

For any $i = 1, \ldots, \phi$,

$$\begin{aligned}
\|\Theta_i(p') - \Theta_i(p'')\| &= \left\| \min_{\substack{1 \le j \le \phi \\ j \ne i}} \left\{\frac{N_j}{N_i+N_j} p'_j\right\} - \min_{\substack{1 \le j \le \phi \\ j \ne i}} \left\{\frac{N_j}{N_i+N_j} p''_j\right\} \right\| \\
&= \left\| \min_{\substack{1 \le j \le \phi \\ j \ne i}} \left\{\frac{N_j}{N_i+N_j} p'_j - \min_{\substack{1 \le j \le \phi \\ j \ne i}} \left\{\frac{N_j}{N_i+N_j} p''_j\right\}\right\} \right\| \\
&= \left\| \min_{\substack{1 \le j \le \phi \\ j \ne i}} \left\{\frac{N_j}{N_i+N_j} p'_j - \frac{N_k}{N_i+N_k} p''_k\right\} \right\|
\end{aligned}$$

$$\text{where } \frac{N_k}{N_i+N_k}p''_k = \min_{\substack{1\le j\le\phi \\ j\ne i}} \left\{ \frac{N_j}{N_i+N_j}p''_j \right\}.$$

$$\le \left\| \frac{N_k}{N_i+N_k}(p'_k - p''_k) \right\| = \frac{N_k}{N_i+N_k} \left\| p'_k - p''_k \right\|.$$

Therefore,

$$d\left(\Theta(p'),\Theta(p'')\right) = \max_{1\le i\le\phi} \left\|\Theta_i(p') - \Theta_i(p'')\right\| \le \max_{\substack{1\le i\le\phi \\ 1\le k\le\phi \\ i\ne k}} \left\{ \frac{N_k}{N_i+N_k} \left\| p'_k - p''_k \right\| \right\}.$$

But $N_i > 0$ and $N_k > 0$ so $N_k/(N_i + N_k) < 1$ for all $i, k = 1, \ldots, \phi$; $i \ne k$. Hence,

$$d(\Theta(p'),\Theta(p'')) < \max\left\{\|p'_1 - p''_1\|, \ldots, \|p'_\phi - p''_\phi\|\right\} = d(p', p'').$$

■

It follows immediately from the Banach's Fixed-Point Theorem that Θ possesses a unique fixed-point $p^U \in \mathbb{R}^\phi_+$. Hence,,

Corollary 1

The Undercut-Proof equilibrium exists, is unique and is p^U.

3.3 Characterization of the Undercut-Proof Equilibrium

Banach's Fixed-Point Theorem provides an iterative algorithm for computing to an arbitrarily close approximation the fixed point p^U for the mapping Θ. But there is an alternative simple two-step algorithm which computes p^U exactly and which is considerably more informative.

When all firms have the same number of brand-oriented consumers, *i.e.*, equation (7) is satisfied with equalities only ($N_i = N$ for all i), Defintion 2 implies that $p^U_i N = (p_j - T)(2N)$ for all $i, j = 1, \ldots, \phi$, $i \ne j$. Hence, the UPE prices are $p_i = 2T$ for all i.

It is very unlikely in reality that equal numbers of customers will most prefer each product brand. We therefore present a simple algorithm for solving for the Undercut-Proof Equilibrium under the most plausible assumption that (7) is satisfied with strict inequalities.[2]

[2]Algorithms for the case where (7) is satisfied with some strict inequalities and some equalities are very similar to the algorithm for the strict inequality case, but may require some extra notation since in Step II of this algorithm more than one RHS term may achieve the minimum.

Assumption 1

$N_1 < N_2 < \cdots < N_\phi$.

The algorithm involves two steps:

Step I: Solve for the prices of each firm $i = 2, 3, \ldots, \phi$, as functions of p_1. That is, this algorithm asserts that each firm $i = 2, 3, \ldots, \phi$ sets its price under the assumption that firm 1 is the most likely to undercut its price. Formally, set

$$p_i = T + \frac{N_1}{N_i + N_1} p_1, \quad \text{for all firms } i = 2, 3, \ldots, \phi.$$

Step II: Let firm 1 set its price so that no firm $i = 2, 3, \ldots, \phi$ would profit from undercutting firm 1. Formally, set

$$\begin{aligned} p_1 &= T + \min\left\{ \frac{N_2}{N_1 + N_2} p_2, \cdots, \frac{N_\phi}{N_1 + N_\phi} p_\phi \right\} \\ &= T + \min\left\{ \frac{N_2}{N_1 + N_2}\left(T + \frac{N_1}{N_2 + N_1} p_1\right), \cdots, \frac{N_\phi}{N_1 + N_\phi}\left(T + \frac{N_1}{N_\phi + N_1} p_1\right) \right\}. \end{aligned}$$

Theorem 1

The Undercut-Proof equilibrium p^U is the unique solution to the above algorithm.

Proof. To prove Step I we need to show that in equilibrium, if firm 1 cannot strictly increase its profit by undercutting an arbitrary firm k, $k \neq 1$, then no other firm ℓ, $\ell \neq 1$ and $\ell \neq k$, can strictly enhance its profit by undercutting firm k. Intuitively speaking, we show that if any firm k keeps its price low enough so firm 1 will not find it profitable to undercut, then firm k is also 'safe' from being undercut by any firm $\ell \neq k$.

By a way of contradiction suppose not. Then, there is a firm ℓ for which

$$\begin{aligned} (p_k^U - T)(N_1 + N_k) &\leq p_1^U N_1 \quad \text{(firm 1 does not undercut } k) \\ (p_k^U - T)(N_\ell + N_k) &> p_\ell^U N_\ell \quad \text{(firm } \ell \text{ undercuts } k) \end{aligned}$$

Subtracting the first equation from the second,

$$(p_k^U - T)(N_\ell - N_1) > p_\ell^U N_\ell - p_1^U N_1 \geq \min\{p_1^U, p_\ell^U\}(N_\ell - N_1).$$

Hence, since $N_1 < N_\ell$,

$$p_k^U - T > \min\{p_1^U, p_\ell^U\}$$

implying that firm k is being undercut; hence, p^U is not an equilibrium price. A contradiction.

Finally,

$$p_1^U = T + \min\left\{\frac{N_2}{N_1+N_2}p_2^U, \cdots, \frac{N_\phi}{N_1+N_\phi}p_\phi^U\right\},$$

which is identical to step II of our algorithm. Therefore, p^U solves the algorithm. But, clearly, the solution to our algorithm is unique so p^U must be the only solution to our algorithm. ■

The intuition behind Theorem 1 is that each firm $i \geq 2$ is concerned that it will be undercut by firm 1 and therefore adjusts its price so firm 1 will not undercut. As it turns out, if firm 1 does not gain from undercutting any firm $i \geq 2$, then no other firm can benefit from undercutting firm i. Finally, firm 1 sets its price so no other firm will undercut firm 1.

3.4 Properties of the Undercut-Proof Equilibrium

We first analyze the relationship between consumer orientation towards the brands and equilibrium prices.

Theorem 2

The firm with the lowest market share charges the highest price, the firm with the second lowest market share charges the second highest market price, and so on. Formally, in an UPE, *if* $N_1 < N_2 < \cdots < N_\phi$ *then*

$$p_1^U > p_2^U > \cdots > p_\phi^U.$$

Proof. For $i = 2, 3, \ldots, \phi$,

$$p_i^U = T + \frac{N_1}{N_i + N_1}p_1^U.$$

Since $N_2 < \cdots < N_\phi$, we have that $p_2^U > \cdots > p_\phi^U$.

It remains to show that $p_1^U > p_2^U$. There exists a $j \in \{2, 3, \ldots, \phi\}$ for which $p_1^U = T + N_j p_j^U/(N_1 + N_j)$. Since $p_2^U = T + N_1 p_1^U/(N_2 + N_1)$, we obtain

$$p_1^U = \frac{(N_1+N_j)(N_1+2N_j)}{(N_1)^2 + N_1N_2 + (N_2)^2}T \quad \text{and} \quad p_2^U = \left[1 + \frac{N_1(N_1+N_j)(N_1+2N_j)}{(N_2+N_1)((N_1)^2+N_1N_j+(N_j)^2)}\right]T.$$

Therefore $p_1^U - p_2^U =$

$$\frac{(N_2+N_1)\left[(N_1+N_j)(N_1+2N_j)-(N_1)^2-N_1N_j-(N_j)^2\right]-N_1(N_1+N_j)(N_1+2N_j)}{(N_2+N_1)\left[(N_1)^2+N_1N_j+(N_j)^2\right]}T. \tag{11}$$

(11) verifies that $p_1^U - p_2^U > 0$. ■

The next proposition analyzes the relationship between consumer orientation towards the brands and equilibrium profit levels. The proof is provided in Appendix B.

Theorem 3

The firm with the lowest market share makes the lowest profit, the firm with the second lowest market share makes the second lowest profit, and so on. Formally, in an UPE*, if* $N_1 < N_2 < \cdots < N_\phi$ *then*

$$\pi_1^U < \pi_2^U < \cdots < \pi_\phi^U.$$

We conclude this section by demonstrating four additional properties of the UPE. Let $N \stackrel{\text{def}}{=} (N_1, \ldots, N_\phi)$. Then

1. UPE prices converge to zero when the brands become less differentiated. Formally, as $T \to 0$, $p^U(N,T) \to \vec{0}$.

2. $p^U(N,T)$ is homogeneous of degree zero in N and is homogeneous of degree one in T.

3. Let $\pi^U \stackrel{\text{def}}{=} \left(\pi_1^U(N,T), \ldots, \pi_\phi^U(N,T)\right)$. Then, π^U is homogeneous of degree one in N and is homogeneous of degree one in T.

4. Adding in a new firm with N_k oriented customers affects neither the price nor the profit of any incumbent firm provided that

$$(N_k - N_j)\left[N_1N_j - N_kN_j + 2(N_1)^2 + N_1N_k\right] > 0,$$

where j is the firm jointly determining p_1. Otherwise the addition of the new firm reduces the UPE prices and profit levels of all incumbent firms.

We find the last property to be quite remarkable, since it states that if a newly entering firm is not a threat to firm 1, then entry will not affect market prices. In contrast, if the entering firm is a threat to firm 1, then entry will result in lower prices for all incumbent firms. Thus, the effect of entry on prices depends on whether the entrant is a threat for the firm with the lowest market share.

4. Examples

We now demonstrate the ease of using the algorithm for computing the Undercut-Proof equilibrium in simple examples.

4.1 Firm 1 'fears' being undercut by firm 2

Suppose that there are only three firms and that $N_1 = 1$, $N_2 = 2$, and $N_3 = 3$.

Using Step I of the algorithm, firm 2 and firm 3 set prices so that

$$p_2^U = T + \frac{N_1}{N_2 + N_1} p_1^U = T + \frac{1}{3} p_1^U, \quad \text{and} \quad p_3^U = T + \frac{N_1}{N_3 + N_1} p_1^U = T + \frac{1}{4} p_1^U.$$

Using step II of the algorithm, firm 1 sets its price according to

$$\begin{aligned} p_1^U &= T + \min\left\{ \frac{N_2}{N_1 + N_2}\left(T + \frac{1}{3}p_1^U\right), \frac{N_3}{N_1 + N_3}\left(T + \frac{1}{4}p_1^U\right) \right\} \\ &= T + \min\left\{ \frac{2}{3}\left(T + \frac{1}{3}p_1^U\right), \frac{3}{4}\left(T + \frac{1}{4}p_1^U\right) \right\} \\ &= T + \min\left\{ \frac{15T}{7}, \frac{28T}{13} \right\} = \frac{15T}{7}. \end{aligned}$$

Hence, in this example, firm 1 sets its price to prevent being undercut by firm 2, thereby also preventing being undercut by firm 3. Therefore

$$p_1^U = \frac{15T}{7} > p_2^U = \frac{12T}{7} > p_3^U = \frac{43T}{28}, \quad \text{and} \quad \pi_1 = \frac{15T}{7} < \pi_2 = \frac{24T}{7} < \pi_3 = \frac{129T}{28}.$$

4.2 Firm 1 'fears' being undercut by firm 3

Suppose now that $N_1 = 1$, $N_2 = 2$, and $N_3 = 10$.

Using Step I of the algorithm, firm 2 and firm 3 set prices so that

$$p_2^U = T + \frac{N_1}{N_2 + N_1} p_1^U = T + \frac{1}{3} p_1^U, \quad \text{and} \quad p_3^U = T + \frac{N_1}{N_3 + N_1} p_1^U = T + \frac{1}{11} p_1^U.$$

Using step II of the algorithm, firm 1 sets its price according to

$$\begin{aligned} p_1^U &= T + \min\left\{ \frac{N_2}{N_1 + N_2}\left(T + \frac{1}{3} p_1^U\right), \frac{N_3}{N_1 + N_3}\left(T + \frac{1}{11} p_1^U\right)\right\} \\ &= T + \min\left\{ \frac{15T}{7}, \frac{77T}{37} \right\} = \frac{77T}{37}. \end{aligned}$$

Hence, in this example, firm 1 sets its price to prevent being undercut by firm 3, thereby also preventing being undercut by firm 2. Therefore

$$p_1^U = \frac{77T}{37} > p_2^U = \frac{188T}{111} > p_3^U = \frac{44T}{37}, \quad \text{and} \quad \pi_1 = \frac{77T}{37} < \pi_2 = \frac{376T}{111} < \pi_3 = \frac{440T}{37}.$$

4.3 Undercut-Proof equilibrium applied to the Hotelling model

So far we have applied the UPE to models in which Nash-Bertrand equilibria do not exist. A natural question to ask is how the UPE and the Nash-Bertrand equilibrium compare in environments when a Nash-Bertrand equilibrium does exist.

4.3.1 Genral formulation

Consider the Hotelling (1929) linear city with a length of one unit and a continuum of consumers uniformly distributed with unit density. There are two firms, A and B, located at a distances of a and b from the edges of the town respectively Figure 2 illustrates this city.

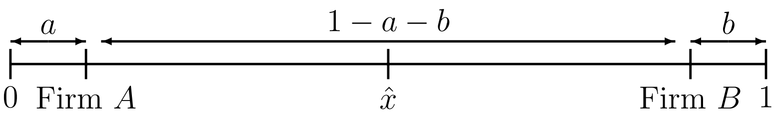

Figure 2: Hotelling's linear city

Let $\tau > 0$ denote the per-unit-of-distance consumers' transportation cost. The utility of a consumer located at point x is

$$U_x \stackrel{\text{def}}{=} \begin{cases} -p_A - \tau|x - a| & \text{if she buys from } A \\ -p_B - \tau|1 - b - x| & \text{if she buys from } B. \end{cases}$$

The location of the consumer who is 'indifferent' between buying brand A and B is denoted by $\hat{x}$ and is defined implicitly by $-p_A - \tau(\hat{x} - a) = -p_B - \tau(1 - b - \hat{x})$. Hence,

$$\hat{x}(p_A, p_B) = \frac{1}{2} + \frac{\tau(a-b) + p_B - p_A}{2\tau}. \tag{12}$$

The profit of firm A is $\pi_A = p_A x(p_A, p_B)$ and of firm B is $\pi_B = p_B[1 - x(p_A, p_B)]$. The UPE is characterized by

$$\begin{aligned} \pi_B^U = p_B^U \left[\hat{x}(p_A, p_B)\right] \quad &\geq \quad 1 \times \left[p_A^U - \tau(1 - a - b)\right] \\ \pi_A^U = p_A^U \left[1 - \hat{x}(p_A, p_B)\right] \quad &\geq \quad 1 \times \left[p_B^U - \tau(1 - a - b)\right]. \end{aligned} \tag{13}$$

Substituting (13) into (12) yields

$$\hat{x} = \frac{\hat{x}^2(a - b + 1) - \hat{x}(3a + b - 1) + 2a}{2(\hat{x}^2 - x + 1)}. \tag{14}$$

The proof of the following proposition is given in Appendix C

Proposition 4

1. *There exists a UPE in the general Hotelling model which is a solution of (12) and (13).*
2. *In case of multiple equilibria, the stable UPE is unique.*

Figure 3 illustrates how an equilibrium is determined.

PETER, WE NEED TO PLACE HERE SOME CHARACTERIZING PROPOSITIONS LIKE: (I AM NOT SURE THAT THESE ARE CORRECT!)

Proposition 5

Under the stable UPE,

1. *The firm located closer to the center has a larger market share. Formally, $x^* \geq 1/2$ if $a \geq b$.*
2. *The firm with the larger market share charges a lower price (but makes a higher profit). Formally, $p_A^U \leq p_B^U$ (and $\pi_A^U \geq \pi_B^U$) if and only if $x^* \geq 1/2$.*

*** BELOW IS THE SYMMETRIC CASE FROM THE PREVIOUS VERSION ***

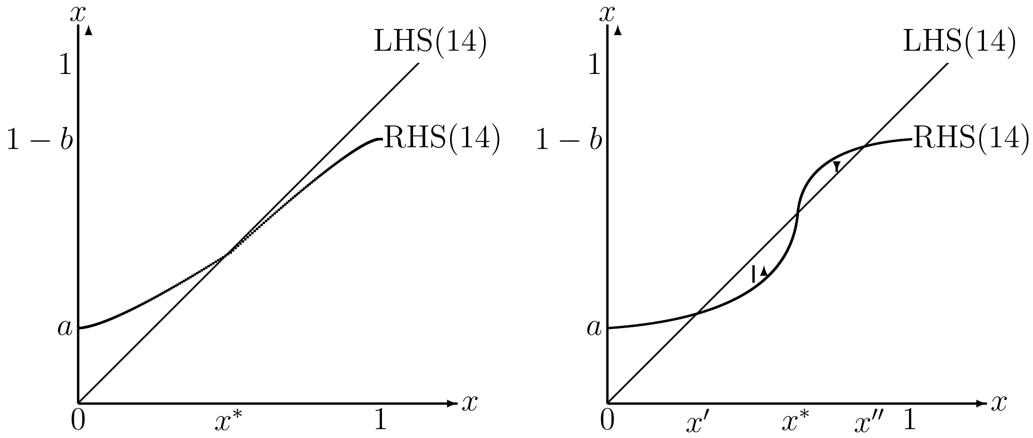

Figure 3: *Left:* Unique equilibrium. *Right:* Multiple equilibria (one stable and two unstable).

4.3.2 Special case: Hotelling with symmetric locations

The derivation of the Nash-Bertrand equilibrium in the Hotelling model is straight forward, and we therefore only state that a Nash-Bertrand equilibrium does not exist when $1/4 < d < 1/2$. A Nash-Bertrand equilibrium exists only if the firms are located sufficiently far from each other. More precisely, the Nash-Bertrand equilibrium prices exist only if $0 \le d \le 1/4$ or $d = 1/2$. If $d = 1/2$, the brands are homogeneous and therefore the Nash equilibrium prices and profit levels are zero. For $0 \le d \le 1/4$,

$$p_A^N = p_B^N = \tau, \quad \text{and} \quad \pi_A^N = p_B^N = \frac{\tau}{2}. \tag{15}$$

Figure 4 compares the two equilibria given in (**??**) and (15).

The UPE always exists even if the firms are located close to each other. In contrast, a Nash-Bertrand equilibrium does not exist when $1/4 < d < 1/2$ precisely because firms benefit from undercutting their competitors' prices. Moreover, the UPE prices monotonically decline when the firms locate closer to each other ($d \to 1/2$). In contrast, at the distances when a Nash-Bertrand equilibrium exists, the Nash-Bertrand equilibrium prices *do not vary* with the distance between the firms. We view this property of the Nash-Bertrand prices as unreasonable. Finally, when the firms are far from each other, the UPE prices exceed the Nash-Bertrand prices. However, at the distances where a Nash-Bertrand equilibrium ceases to exist (*i.e.*, $d = 1/4$ and $d = 1/2$) the two equilibrium price pairs coincide.

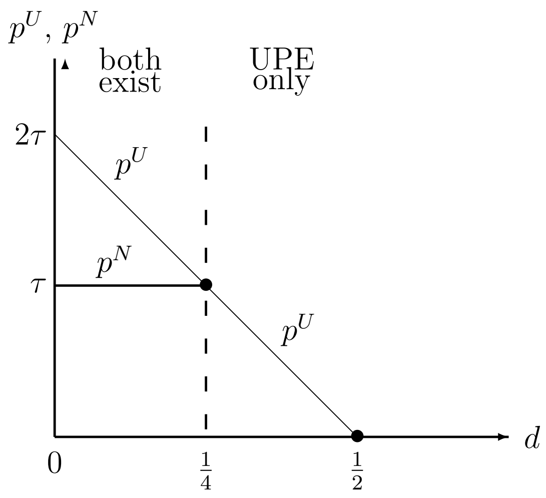

Figure 4: Comparing the UPE with the Nash-Bertrand equilibrium

5. Some Dynamic Justifications for the UPE Concept

In this section we demonstrate the role played by the UPE prices in infinite-horizon models of price competition. We present two examples from Industrial Organization. The first involves the well known *meeting-the-competition* game where stores adhere to previously advertised prices, or match a rival's store price when the rival store offers a discount. The second dynamic example involves a relationship between suppliers and retailers via *resale-price maintenance* where a manufacturer sets price ceilings for each one of his dealers.

5.1 Meeting-the-competition clause (MCC)

We demonstrate that the UPE prices serve as upper bounds on colluding prices in a *meeting-the-competition* game (see for example, Dixit and Nalebuff 1991, p.103; and Salop 1986). In this game, consumers purchase from the (transportation-cost inclusive) lowest price and in addition, they do not tolerate price increases. We demonstrate that in a SPE, stores credibly commit to match the price of the rival store if the rival store undercuts its price.

Consider an infinite-horizon discrete-time economy and the two competing stores described in Section 2. In each period t, $(t = 1, 2, \ldots)$, N_α A-oriented consumers and N_β B-oriented consumers enter the market and purchase at most one unit from one store. Let

p_A^t and p_B^t denote the period t stores' prices. We now modify the preferences given in (1) by adding reservation utility for consuming each brand.

$$U_\alpha \stackrel{\text{def}}{=} \begin{cases} V_A^t - p_A^t & \text{buying from } A \\ V_B^t - p_B^t - T & \text{buying from } B \end{cases} \quad \text{and} \quad U_\beta \stackrel{\text{def}}{=} \begin{cases} V_A^t - p_A - T & \text{buying from } A \\ V_B^t - p_B & \text{buying from } B. \end{cases} \tag{16}$$

The following assumption implies that consumers do not tolerate price increases.

Assumption 2

Consumers' willingness to pay for each brand does not increase over time and decreases with the last price paid. Formally,

$$V_A^t \stackrel{\text{def}}{=} \min\{V_A^{t-1}, p_A^{t-1}\} \quad \textit{and} \quad V_B^t \stackrel{\text{def}}{=} \min\{V_B^{t-1}, p_B^{t-1}\}. \tag{17}$$

Whereas from a technical point of view the validity of this procedure is debatable (since the fundamentals of the model are redefined by state variables) Assumption 2 serves our purpose for this illustration of the dynamic properties of the UPE concept. However, it should be mentioned that Assumption 2 has a tremendous amount of empirical support. For example, Gabor (1988, Ch.11) demonstrates the importance consumers attach to the last-price paid in their determination of the maximum willingness to pay. From a theoretical point of view Assumption 2 can be justified by assuming that the price signals the quality of the product. Finally, this assumption could be justified by modeling strategic consumers who punish a store that raises the price.

All stores have the same constant marginal cost of production, normalized to zero, and all have the same periodic discount factor $0 < \delta < 1$. At any date t, store i's one-period profit is $\pi_i(p_A^t, p_B^t)$. Let $p^t = (p_A^t, p_B^t) \in \mathbb{R}_+^2$ be the vector of prices in period t. The vector of stores' period t profits is $\pi(p^t) = (\pi_A(p_A^t, p_B^t), \pi_B(p_A^t, p_B^t))$. We will also use π_i^t to denote $\pi_i(p_1^t, p_2^t)$.

For a given sequence of price vectors $\langle p^{t+s}\rangle_{s=0}^{\infty}$, store i's present-valued profit at date t is $\sum_{s=0}^{\infty} \delta^s \pi_i(p_A^{t+s}, p_B^{t+s})$. Each store maximizes its own present-valued profit.

In this alternating-moves price-setting game, store A sets its price in odd periods $t = 1, 3, 5, \dots$, and store B sets its price in even periods $t = 2, 4, 6, \dots$. Each store is committed to maintaining its price for two periods. Hence, $p_A^{2k} = p_A^{2k-1}$, and $p_B^{2k+1} = p_B^{2k}$ for all $k = 1, 2, 3, \ldots$.

It is assumed that store i's pricing decision for period t depends only upon prices which prevailed in period $t-1$. This is a Markovian assumption which makes the dynamic best-response function R_i of any store i dependent only upon the price committed by its rival store in the previous period and itself, so that $p_i^t = R_i(p^{t-1})$, $i = A, B$; $i \neq j$, see Maskin and Tirole (1987, 1988a,b), or Eaton and Engers (1990) who model differentiated products which in fact resembles very much the present framework.

In a *meeting-the-competition* game, each store states that it would match the (transportation-cost inclusive) price of its rival store, whenever the rival store undercuts. Formally, this game is characterized by the following strategy functions.

DEFINITION 3

The dynamic functions are called ***meeting-the-competition*** *response functions if for every period t in which store i is entitled to set its price,*

$$p_i^t = R_i(p_A^{t-1}, p_B^{t-1}) \stackrel{\text{def}}{=} \begin{cases} p_i^{t-1} & \text{if } p_j^{t-1} \geq p_i^{t-1} - T \\ p_j^{t-1} + T & \text{if } p_j^{t-1} < p_i^{t-1} - T \end{cases} \qquad i, j = A, B, \; i \neq j. \tag{18}$$

Thus, a store will not alter its price, unless the other store undercut it in an earlier period. If the competing store undercuts, the store matches the reduced (transportation-inclusive) price in a subsequent period. Notice that when the transportation cost parameter is small ($T \to 0$), stores respond to any price reduction of the rival store, meaning that the competition is always met for any price reduction (which is an assumption made also in Kalai and Satterthwaite 1994 in a standard static Bertrand environment). If T is large, we have differentiated-brands markets which allow stores' prices to differ.

Next, define the dynamically-modified UPE prices by

$$p_A^U(\delta) \stackrel{\text{def}}{=} \frac{[N_\alpha + (1-\delta)N_\beta]\,[(1-\delta)N_\alpha + 2N_\beta]\,T}{(1-\delta)\,[(N_\alpha)^2 + (1-\delta)N_\alpha N_\beta + (N_\beta)^2]} \tag{19}$$

and

$$p_B^U(\delta) \stackrel{\text{def}}{=} \frac{[(1-\delta)N_\alpha + N_\beta]\,[2N_\alpha + (1-\delta)N_\beta]\,T}{(1-\delta)\,[(N_\alpha)^2 + (1-\delta)N_\alpha N_\beta + (N_\beta)^2]}.$$

Clearly, $p_A^U(\delta) \to p_A^U$ and $p_B^U(\delta) \to p_B^U$ as $\delta \to 0$, where p_A^U and p_B^U are given in (3). Thus, the dynamically-modified UPE prices converge to the static UPE prices as the discount-rate parameter declines to zero.

Proposition 6

Let p_A^0 and p_B^0 be given. Then, the meeting-the-competition response functions (18) constitute a SPE if and only if $p_A^0 \le p_A^U(\delta)$ and $p_B^0 \le p_B^U(\delta)$.

Proof. Observe that on the equilibrium path, no store reduces its price. If one store deviates and reduces it price, then equilibrium is restored at a reduced prices of all stores. In particular, in a SPE no store can increase its profit by once undercutting its rival store. Hence, for (18) to constitute a SPE, it must be that for every odd t,

$$(N_\alpha + N_\beta)(p_B^{t-1} - T) + \delta N_\alpha \frac{p_B^{t-1} - T}{1-\delta} \le N_\alpha \frac{p_A^{t-1}}{1-\delta}, \text{ or } p_A^{t-1} \ge \frac{[N_\alpha + (1-\delta)N_\beta](p_B^{t-1} - T)}{N_\alpha}; \tag{20}$$

and for every even t,

$$(N_\alpha + N_\beta)(p_A^{t-1} - T) + \delta N_\beta \frac{p_A^{t-1} - T}{1-\delta} \le N_\beta \frac{p_B^{t-1}}{1-\delta}, \text{ or } p_B^{t-1} \ge \frac{[(1-\delta)N_\alpha + N_\beta](p_A^{t-1} - T)}{N_\beta}. \tag{21}$$

Figure 5 illustrates the regions determined by (20) and (21) in the period $t-1$ price space.

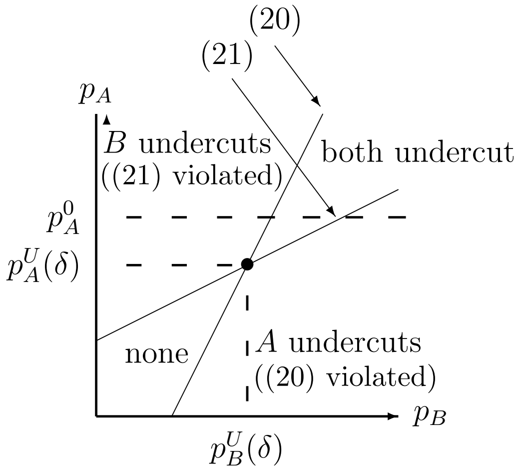

Figure 5: Undercutting regions in a dynamic meeting-the-competition game

Suppose that (18) constitutes a SPE, but that either $p_A^0 > p_A^U(\delta)$ or $p_B^0 > p_B^U(\delta)$. With no loss of generality, suppose that $p_A^0 > p_A^U(\delta)$. Figure 5 shows that either (20) or (21) must be violated for any value of p_B^0. A contradiction.

To demonstrate the second part of the proposition, suppose that $p_A^0 \leq p_A^U(\delta)$ and $p_B^0 \leq p_B^U(\delta)$. We now show that (18) constitutes a SPE. Clearly, no store would benefit from reducing its price by less than T since this would reduce the profit of the undercutting store forever without enlarging its market share. Also, since on the equilibrium path $p_A^t \leq p_A^U(\delta)$ and $p_B^t \leq p_B^U(\delta)$, no store can increase its profit from one period undercutting. Finally, stores cannot increase profit by increasing price since Assumption 2 implies that a store's profit declines to zero when it raises its price. ■

The meeting-the-competition clause is regarded as an implicit collusion mechanism. Therefore, Proposition 6 can be interpreted as follows. Suppose that before the game starts, the firms collude on prices p_A^0 and p_B^0, and only later enter the meeting-the-competition stage. Then,

Corollary 2

Under the meeting-the-competition collusion mechanism, the collusion prices cannot exceed the UPE prices given in (19).

5.2 Resale price maintenance (RPM)

In this game we separate the stores from a single manufacturer (supplier) who, for some reason, cannot sell directly to consumers, and therefore sells the product to two stores (dealerships). As before, we assume that the (consumer-invariant) transportation cost between the stores is \$$T$, where N_α consumers locating near store A, and N_β consumers locating near store B. This game differs from the previous example, subsection 5.1, in two respects: First, in this game consumers do not have reservation prices. Secondly, the dynamic UPE prices differ from the MCC game for all $\delta > 0$, however, similar to the MCC example, they also converge to the static UPE prices when $\delta \to 0$.

We assume that the manufacturer can enforce a *price ceiling* on each dealer, and that there is no other mean or contract that can be written between the supplier and the dealer.

Finally, as in any dealership problem, we assume that the manufacturer's goal is to maintain the highest possible price under the restriction that the dealers would not find it profitable to engage in price reduction. Obviously, enforcing a price floor (instead of a ceiling) could prevent price undercutting, but for our purpose, we assume that setting price floors is illegal.

Define the new dynamic UPE prices by

$$p_A^U \stackrel{\text{def}}{=} \frac{[N_\alpha + 2(1+\delta)N_\beta][(1+\delta)N_\alpha + N_\beta]T}{(1+\delta)[(N_\alpha)^2 + (N_\beta)^2] + N_\alpha N_\beta} \tag{22}$$

and

$$p_B^U \stackrel{\text{def}}{=} \frac{[2(1+\delta)N_\alpha + N_\beta][N_\alpha + (1+\delta)N_\beta]T}{(1+\delta)[(N_\alpha)^2 + (N_\beta)^2] + N_\alpha N_\beta}.$$

Note that despite the fact that these UPE prices are different from the previous example given in (19), we also have that $p_A^U(\delta) \to p_A^U$ and $p_B^U(\delta) \to p_B^U$ as $\delta \to 0$, where p_A^U and p_B^U are given in (3).

For this alternating-price-setting game we define the resale-price maintenance game as follows.

Definition 4

*Let the manufacturer assign $\bar{p}_A$ and $\bar{p}_B$ as the price ceiling to dealer A and dealer B respectively. Then, the **the resale-price-maintenance** response functions are given by*

$$p_i^t = R_i(p_A^{t-1}, p_B^{t-1}) \stackrel{\text{def}}{=} \begin{cases} \bar{p}_i & \text{if } p_j^{t-1} \geq \bar{p}_i - T \\ p_j^{t-1} + T & \text{if } p_j^{t-1} < \bar{p}_i - T \end{cases} \qquad i, j = A, B,\ i \neq j. \tag{23}$$

That is, each store i sets the price as close as possible to its RPM price, $\bar{p}_i$ subject to the constraint that it keeps a strictly positive market share. Hence, if the price set by its rival, store j, in a previous period is very low, store i will set the highest price subject not to be undercut; hence $p_i^t = p_j^{t-1} + T$.

Proposition 7

Let $p_i^U(\delta)$ be given, and $\bar{p}_i$ be given in (23), $i = A, B$. Then, the resale-price-maintenance response functions given in (23) constitute a SPE if and only if $\bar{p}_i \leq p_i^U(\delta)$.

Proof. On the equilibrium path, each store sets its RPM price set by the manufacturer. Also, the response functions (23) imply that a price reduction by one of the stores is followed by either change in the price set by the other store, or, stores gradually raise prices until they hit the RPM level. Clearly, if store i undercuts store j, it cannot benefit from setting its price any lower than $p_j^{t-1} - T$. Hence, for (23) to constitute a SPE, it must be that for every odd t,

$$(N_\alpha + N_\beta)(\bar{p}_B - T) + \delta N_\alpha(\bar{p}_B - T) + \frac{\delta^2}{1-\delta} N_\alpha \bar{p}_A \leq N_\alpha \frac{\bar{p}_A}{1-\delta}; \tag{24}$$

and for every even t,

$$(N_\alpha + N_\beta)(\bar{p}_A - T) + \delta N_\alpha(\bar{p}_A - T) + \frac{\delta^2}{1-\delta} N_\alpha \bar{p}_B \leq N_\alpha \frac{\bar{p}_B}{1-\delta}; \tag{25}$$

Clearly, by construction, if $\bar{p}_i > p_i^U(\delta)$ for at least one store i, then either (24) or (25) must be violated.

To demonstrate the second part of the proposition, note that if $\bar{p}_i \leq p_i^U(\delta)$ for $i = A, B$, then no store can increase its profit by lowering its price below its RPM set level. ■

6. Discussion

The equilibrium concept developed in this paper can be applied to a wide variety of market games involving price-setting brand-producing firms competing in markets characterized by heterogeneous consumers, discrete location-address models, or homogeneous products models with capacity constraints, and is by no means restricted to analyzing differentiated products only.

The Undercut-Proof equilibrium specifies a specific type of conjectural variations behavior in which each firm assumes that the rival firm will alter its price *only* if such an action satisfies two properties: (a) the undercutting firm will enlarge its market share by appropriating the customers of the firms it undercuts, and (b) such undercutting is profitable.

It is often argued that firms selling to a large number of customers prefer to charge a lower price than do firms selling to few consumers, because of increasing returns to scale production technologies. Here we obtain this same price differential result for an entirely different reason and without assuming increasing returns. In an Undercut-Proof equilibrium the firm with the larger market share charges the lower price since it is more vulnerable to being undercut. That is, the larger is the market share of a 'large' firm, the more a firm with a smaller market share can gain by undercutting. Thus, the reason why large firms tend to charge lower prices may be to protect market shares from competitors and need not stem from the existence of increasing returns production technologies.

Appendix A. Proof of equation (11)

Expanding (11) yields

$$\begin{aligned} p_1^U - p_2^U &\propto 2N_1N_2N_j + N_2(N_j)^2 - (N_1)^3 - (N_1)^2N_j - N_1(N_j)^2 \\ &> 2N_1N_2N_j + N_2(N_j)^2 - (N_1)^3 - N_1N_2N_j - N_1(N_j)^2 \quad \boxed{\text{because } N_2 > N_1} \\ &= N_1(N_2N_j - (N_1)^2) + (N_j)^2(N_2 - N_1) > 0. \quad \boxed{\text{because } N_j > N_2 > N_1} \end{aligned}$$

Appendix B. Proof of Theorem 3

$$p_1^U = T + \frac{N_j}{N_1 + N_j} p_j^U \quad \text{for some } j = 2, 3, \ldots, \phi.$$

Since

$$p_j^U = T + \frac{N_1}{N_j + N_1} p_1^U,$$

it follows that

$$\begin{aligned} p_1^U &= \frac{(N_1 + N_j)(N_1 + 2N_j)}{(N_1)^2 + N_1N_j + (N_j)^2} T, \quad \text{and} \\ p_j^U &= \frac{(N_1 + N_j)(2N_1 + N_j)}{(N_1)^2 + N_1N_j + (N_j)^2} T, \quad \text{and} \\ p_k^U &= \frac{2(N_1)^3 + (N_1)^2(N_k + 4N_j) + N_1N_j(N_k + 3N_j) + N_k(N_j)^2}{(N_1 + N_k)\left[(N_1)^2 + N_1N_j + (N_j)^2\right]} T, \end{aligned}$$

for $k = 2, 3, \ldots, \phi$; $k \neq j$. Therefore,

$$\pi_k^U - \pi_j^U = \frac{2(N_1)^3 + (N_1)^2(N_k + 3N_j) + N_1N_j(N_j + N_k) + N_k(N_j)^2}{(N_1 + N_k)\left[(N_1)^2 + N_1N_k + (N_j)^2\right]}(N_k - N_j)T.$$

$N_k > N_j$ then implies $\pi_k^U > \pi_j^U$. Hence, $\pi_2^U < \pi_3^U < \cdots < \pi_\phi^U$.

It remains to show that $\pi_1^U < \pi_2^U$. Suppose $j = 2$. Then,

$$\pi_1^U - \pi_2^U = \frac{(N_1+N_2)^2(N_1-N_2)}{(N_1)^2+N_1N_2+(N_2)^2}T < 0 \quad \text{since} \quad N_1 < N_2.$$

Hence, $\pi_1^U < \pi_2^U$ if $j = 2$.

Suppose instead that $j \geq 3$. Then, $\pi_1^U - \pi_2^U =$

$$\frac{(N_1)^4+(N_1)^3(3N_j-N_2)+(N_1)^2N_j(2N_j-N_2)-N_1N_j\left[N_2N_j+(N_2)^2\right]-(N_2)^2\left[(N_1)^2+(N_j)^2\right]}{(N_1+N_2)\left[(N_1)^2+N_1N_j+(N_j)^2\right]}T. \tag{26}$$

Also, since

$$p_1^U = T + \frac{N_j}{N_1+N_j}p_j^U, \quad \text{necessarily} \quad \frac{N_2}{N_1+N_2}p_2^U \geq \frac{N_j}{N_1+N_j}p_j^U.$$

Substituting and rearranging shows that this inequality implies

$$\frac{N_1\left[2(N_1)^2+N_1(N_2+N_j)-N_2N_j\right]}{(N_1+N_2)^2\left[(N_1)^2+N_1N_j+(N_j)^2\right]}(N_2-N_j)T \geq 0,$$

which, since $N_2 < N_j$, implies that $2(N_1)^2 + N_1(N_2+N_j) - N_2N_j \leq 0$, from which we obtain that

$$N_2N_j \geq N_1(N_j + 2N_1 + N_2). \tag{27}$$

The numerator of (26) is

$$\begin{aligned}
(N_1)^4 + 3(N_1)^3N_j \;&+\; 2(N_1)^2(N_j)^2 - (N_1)^3N_2 - (N_1)^2(N_2)^2 - N_2N_j\left[N_1N_j + (N_1)^2 - N_1N_2 + N_2N_j\right] \\
&\leq\; (N_1)^4 + 3(N_1)^3N_j + 2(N_1)^2(N_j)^2 - (N_1)^3N_2 - (N_1)^2(N_2)^2 \\
&\qquad - N_1(N_j + 2N_1 + N_2)\left[N_1N_j + (N_1)^2 - N_1N_2 + N_2N_j\right] \\
&=\; N_1\left[(N_j)^2(N_1-N_2) - 4N_1N_2N_j - (N_1)^3 - 4(N_1)^2N_2 - 2N_1(N_2)^2 - (N_2)^2N_j\right] \\
&<\; 0. \quad \boxed{\text{since } N_1 < N_2}
\end{aligned}$$

Hence, $\pi_1^U < \pi_2^U$ when $j \geq 3$. ■

Appendix C. Proof of Proposition 4

Note that RHS(14) $= a$ when $x = 0$, whereas RHS(14) $= 1 - b$ when $x = 1$. In addition, RHS(14) is strictly increasing with x for $x \in [0, 1]$. Hence, by continuity, there must exists an x^* that solves (14).

Next, we need to establish that in the case of multiple equilibria, there exist exactly three equilibria. To see this note that

$$\frac{d^2\text{RHS}(14)}{dx^2} = \frac{(1-a-b)(1-2x)(2+x-x^2)}{(x^2-x+1)^3},$$

which is strictly greater than zero for $x \in [0, 1/2)$ and strictly negative for $x \in (1/2, 1]$. Hence, RHS(14) has exacly one inflection point on the interval $[0, 1]$ meaning that (14) has either one or three solutions, see Figure 3.

Finally, to establish uniqueness of the stable equilibrium using Figure 3 divide this figure into four regions: PETER, DO YOU KNOW EASY-SHORT STABILITY CONDITION FOR THIS PROPOSITION. ■

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