Intermediate Microeconomic Theory
Undergraduate Lecture Notes

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Remarks

- Notes prepared during the 1st semester at the University of Haifa, Nov.97–Jan.98 (Tash-Nach)
- For a Syllabus see a separate handout in Hebrew (summarized by the present Table of Content)
- Texts:
  1. Blumental, Levhari, Ofer, & Sheshinski, 1971. *Price Theory.* Academon Press.
  2. Varian H, 1987, *Intermediate Microeconomics,* W.W. Norton
- Lecture is 3×45 minutes (given nonstop once a week

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Part I

Semester א

Lecture 1
Consumer Theory

1.1 Major Issues

- Characterizing the behavior of a competitive optimizing consumer
- Deriving consumer demand functions from utility maximization
- Consumer problem is divided into "objective" problem (the budget — determined by prices and income); and "subjective" problem (preference, or like)
- Major assumption: consumers are competitive (take prices & income as given).
- Why focusing on 2 goods?

1.2 Budget Constraints

- Define the *commodity space* as $\mathbb{R}_+$
- Define a bundle as a pair (x, y).
- $p_x x + p_y y \leq I$ (set) $\Longrightarrow y = \frac{I}{p_y} - \frac{p_x}{p_y} x$
- Drawing: Intercepts $\frac{I}{p_x}$ and $\frac{I}{p_y}$
- Slope $= -\frac{p_x}{p_y}$
- Income changes, price changes
- Effects of pure inflation λp_x, λp_y, and λI

1.3 Preferences and Utility Functions

Let A, B, & C denote bundles (e.g., $A = (x_0, y_0)$, $B = (x_1, y_1)$, etc.)

- In economics, a consumer is a preference ordering over bundles (subjective)
- Define a preference ordering of a consumer $\succeq$ as a binary relation on $\mathbb{R}_+$ satisfying:
  (a) *Completeness:* Either $A \succeq B$ or $B \succeq A$ for all $A, B \in \mathbb{R}_+$
  (b) *Transitive:* $A \succeq B$ and $B \succeq C \Longrightarrow A \succeq C$
- In most of our analysis we assume that a consumer's preference ordering satisfies *monotonicity*: where
  $(x, y_0) \succeq (x_0, y_0)$ for all $x \geq x_0$; and
  $(x_0, y) \succeq (x_0, y_0)$ for all $y \geq y_0$; and
  $(x, y) \succ (x_0, y_0)$ for all $x > x_0$ and $y > y_0$

- Theorem: Under the 3 axioms, there exists a function U, called *utility function*, $U : \mathbb{R}_+ \to \mathbb{R}$ satisfying
$$(x_0, y_0) \succeq (x_1, y_1) \quad \text{if and only if} \quad U(x_0, y_0) \succeq U(x_1, y_1)$$

1.4 Indifference Curves

- An *indifference curve* for a given utility level U_0 is the set of all bundles, $(x, y) \in \mathbb{R}_+$ yielding $U(x, y) = U_0$
- Properties of indifference curves:
  (a) Never intersect
  (b) Monotonicity $\Rightarrow$ downward sloping
- Rate of Commodity Substitution:
$$\text{RCS}(x, y) = S_{y,x} \stackrel{\text{def}}{=} \left.\frac{dy}{dx}\right|_{U^0}$$
- Marginal Utility:
$$\text{MU}(x, y)_x \stackrel{\text{def}}{=} \frac{\partial U(x, y)}{\partial x} \quad \text{and} \quad \text{MU}(x, y)_y \stackrel{\text{def}}{=} \frac{\partial U(x, y)}{\partial y}$$
- Proposition:
$$\text{RCS}(x, y) = -\frac{\text{MU}(x, y)_x}{\text{MU}(x, y)_y}$$
Proof 1: Define the implicit function $F(x, y(x) \stackrel{\text{def}}{=} U(x, y) - U_0 = 0$. Then, by the implicit function Theorem
$$\frac{\partial y(x)}{\partial x} = -\frac{\partial F(x, y))}{\partial x} = -\frac{\frac{\partial U(x,y)}{\partial x}}{\frac{\partial U(x,y)}{\partial y}} = -\frac{\text{MU}_x(x, y)}{\text{MU}_y(x, y)}$$
Proof 2: Totally differentiating $U(x, y) = U_0$ yields
$$\text{MU}_x dx + \text{MU}_y dy = dU_o = 0 \implies \frac{dy}{dx} = -\frac{\text{MU}_x(x, y)}{\text{MU}_y(x, y)}$$
- Demonstration of various preferences (utility function)
  1. Non-monotonic:
     (a) Bliss point
     (b) Non-monotonic with respect to y only
  2. Perfect Substitutes: $U(x, y) = \alpha x + \beta y$, $\alpha, \beta > 0$.
     Example: Beer in 1 litter (x), and 2 litter bottles (y) $\implies U = x + 2y \implies y = U_0/2 - x/2$
  3. Perfect Complements: $U(x, y) = \min\{\alpha x, \beta y\}$, $\alpha, \beta > 0$. Example: 5 tires for every car.
  4. Cobb-Douglas: $U(x, y) = x^\alpha y^\beta$, $\alpha, \beta > 0$
  5. Quasi-linear: $U(x, y) = \sqrt{x} + y$

1.5 The consumer's utility maximization problem

For given p_x, p_y, and I, choose x and y to

$$\max_{x,y} U(x,y) \quad \textbf{s.t.} \quad p_x x + p_y y \leq I$$

Proposition: Let (x^*, y^*) denote the utility-maximizing bundle. Then, if $x^* > 0$ and $y^* > 0$, then

$$-\frac{\mathrm{MU}_x(x,y)}{\mathrm{MU}_y(x,y)} = -\frac{p_x}{p_y}$$

Note: Show corner solutions for the case where either $x^* = 0$ or $y^* = 0$. Also, show for $U(x,y) = x^2 + y^2$.

1.5.1 Example with perfect substitutes

$U = x + y$, with $I = 12$, $p_x = 3$, & $p_y = 4$
Solution: $x^* = {}^{I}\!/_{p_x} = 4$, $y^* = 0$.

1.5.2 Example with perfect complements

$U = \min\{x, y\}$. Solve $y = x$ with $3x + 4y = 12$ yielding $x^* = y^* = {}^{12}\!/_{7}$.

1.5.3 Example with Cobb-Douglas

$U = x \times y \Longrightarrow x^* = {}^{I}\!/_{2p_x} = 2$ and $y^* = {}^{I}\!/_{2p_y} = {}^{3}\!/_{2}$.

1.5.4 Example with quasi-linear

$U = \sqrt{x} + y \Longrightarrow {}^{p_x}\!/_{p_y} = {}^{3}\!/_{4} = {}^{1}\!/_{2\sqrt{x}} \Longrightarrow x^* = {}^{4}\!/_{9}$.
To find y^* note that $y^* = {}^{I}\!/_{p_y} - {}^{p_x}\!/_{p_y} x^* = {}^{8}\!/_{3}$.

1.5.5 Taxation: specific tax on x only versus income tax

Question: Given that the government collects the same amount of revenue of \$$G$, which tax would be preferred by the consumer?

Sales tax: Let the government impose an tax of \$$t$ on the sale of each unit of x. Now, consumers pay $p_x + t$ for each unit of x. Let $A = (x_0, y_0)$ be the pre-tax chosen bundle, and $B = (x_1, y_1)$ the chosen after the specific tax on x is imposed.

The budget constraint is now $(p_x + t)x + p_y = I$ (steeper budget constraint), the consumer is worse off compared with A. Figure 1.1 illustrates the chosen bundles.

Income tax Set income tax equal to $G \equiv tx_1$ where x_1 is the amount chosen under the specific tax. Now, the budget constraint is $p_x x + p_y y = I - tx_1$. Hence, $B = (x_1, y_1)$ is also affordable (meaning that the new budget constraint passes through (x_1, y_1) with a flatter slope given by ${}^{-p_x}\!/_{p_y}$.

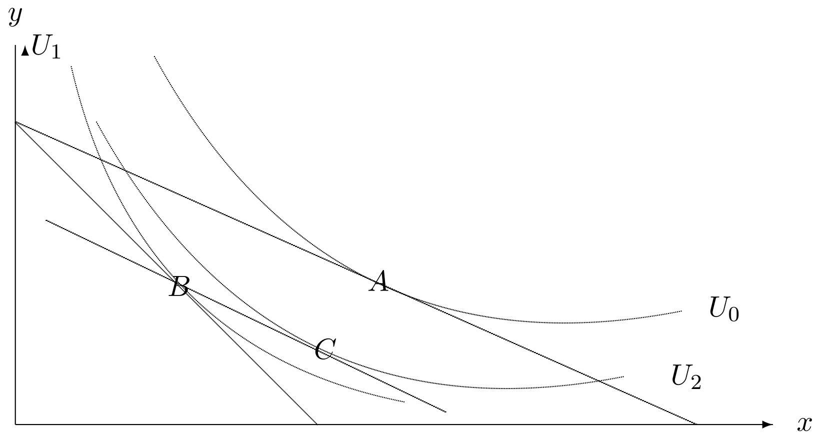

Figure 1.1: Specific tax versus income tax

1.6 Demand functions

Solve

$$\max_{x,y} U(x,y) \quad \textbf{s.t.} \quad p_x x + p_y y \le I$$

to obtain,

$$x(p_x, p_y, I) \quad \text{and} \quad y(p_x, p_y, I)$$

Proposition: Demand functions are homogeneous of degree 0. That is, $x(p_x, p_y, I) = x(\lambda p_x, \lambda p_y, \lambda I)$ for all $\lambda > 0$

proof. No change in the budget constraint. ■

1.6.1 Perfect Substitutes

$U = \alpha x + \beta y$, $\alpha, \beta > 0$, show graphically that

$$x(p_x, p_y, I) = \begin{cases} \frac{I}{p_x} & \text{if } \frac{p_x}{p_y} < \frac{\alpha}{\beta} \\ \text{indeterminate} & \text{if } \frac{p_x}{p_y} = \frac{\alpha}{\beta} \\ 0 & \text{if } \frac{p_x}{p_y} > \frac{\alpha}{\beta} \end{cases} \qquad y(p_x, p_y, I) = \begin{cases} 0 & \text{if } \frac{p_x}{p_y} < \frac{\alpha}{\beta} \\ \text{indeterminate} & \text{if } \frac{p_x}{p_y} = \frac{\alpha}{\beta} \\ \frac{I}{p_y} & \text{if } \frac{p_x}{p_y} > \frac{\alpha}{\beta} \end{cases}$$

1.6.2 Perfect Complements

$U = \min\{\alpha x, \beta y\}$ where $\alpha, \beta > 0$. First equation: $y = (\alpha/\beta)x$. Second equation is the budget constraint.

$$x(p_x, p_y, I) = \frac{\beta I}{\beta p_x + \alpha p_y} \qquad y(p_x, p_y, I) = \frac{\alpha I}{\beta p_x + \alpha p_y}$$

Notice the inverse relationship between quantity demanded and *all* prices!

1.6.3 Cobb-Douglas

$U = x^\alpha \times y^\beta$ where $\alpha, \beta > 0$. Yielding constant-expenditure demand functions:

$$y(p_x, p_y, I) = \frac{\beta I}{(\alpha+\beta)p_y}$$

1.6.4 Quasi-linear

$U = \sqrt{x} + y$. Draw an indifference curve showing the y intercept at U_0 (sloped $-\infty$); and the x intercept at $(U_0)^2$ (sloped $-1/(2U_0)$). Hence, if a corner solution is possible, it must be that $y = 0$.

$$x = \left(\frac{p_y}{2p_x}\right)^2 \quad y = \frac{I}{p_y} - \left(\frac{p_x}{p_y}\right)\left(\frac{p_y}{2p_x}\right)^2 = \frac{I}{p_y} - \frac{p_y}{4p_x} \quad \text{if } \frac{p_x}{p_y} \ge \frac{1}{2U_0}$$

and $x = I/p_x$ (with $y = 0$) otherwise.

1.7 Elasticity

Formally, we define the demand price elasticity by

$$\eta_{x,p_x} \equiv \frac{\partial x(p_x)}{\partial p_x}\frac{p_x}{x}. \tag{1.1}$$

DEFINITION 1.1 *At a given quantity level x, the demand is called*

1. **elastic** *if $\eta_{x,p_x} < -1$ (or, $|\eta_{x,p_x}| > 1$),*
2. **inelastic** *if $-1 < \eta_{x,p_x} < 0$, (or, $|\eta_{x,p_x}| < 1$),*
3. *and has a* **unit elasticity** *if $\eta_{x,p_x} = -1$ (or, $|\eta_{x,p_x}| = 1$).*

For example, in the linear case, $\eta_p(Q) = 1 - a/(bQ)$. Hence, the demand has a unit elasticity when $Q = a/(2b)$. Therefore, the demand is elastic when $Q < a/(2b)$ and is inelastic when $Q > a/(2b)$. Figure 1.2 illustrates the elasticity regions for the linear demand case.

For the constant-elasticity demand function $Q(p) = ap^{-\epsilon}$ we have it that $\eta_p = a(-\epsilon)p^{-\epsilon-1}p/(ap^{-\epsilon}) = -\epsilon$. Hence, the elasticity is constant given by the power of the price variable in demand function. If $\epsilon = 1$, this demand function has a unit elasticity at all output levels.

The marginal revenue (or expenditure) function

The inverse demand function shows the maximum amount a consumer is willing to pay per unit of consumption at a given quantity of purchase. The *total-revenue function* shows the amount of revenue collected by sellers, associated with each price-quantity combination. Formally, we define the total-revenue function as the product of the price and quantity: $TR(Q) \equiv p(Q)Q$. For the linear case, $TR(Q) = aQ - bQ^2$, and for the constant elasticity demand, $TR(Q) = a^{1/\epsilon}Q^{1-1/\epsilon}$. Note that a more suitable name for the revenue function would be to call it the total expenditure function since we actually refer to consumer expenditure rather than producers' revenue. That is, consumers' expenditure need not equal producers' revenue, for example, when taxes are levied on

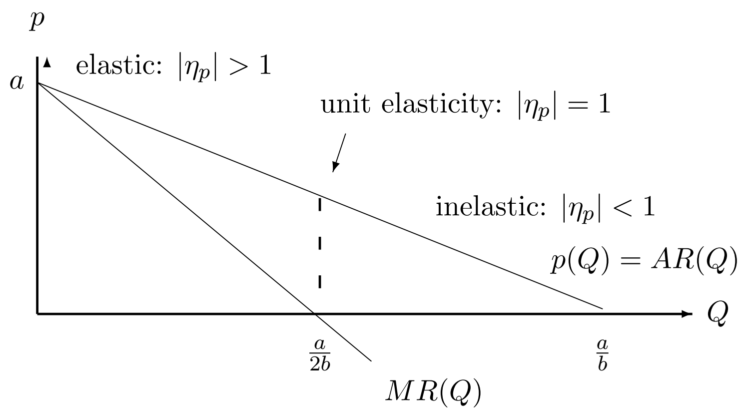

Figure 1.2: Inverse linear demand

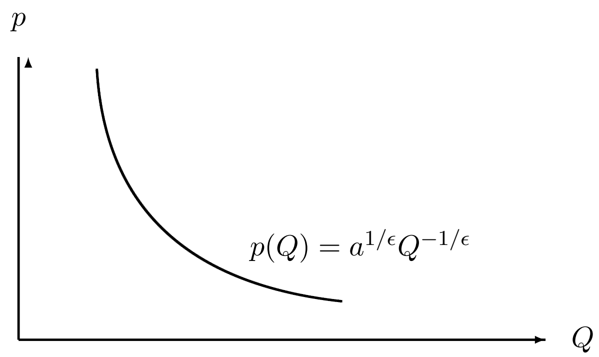

Figure 1.3: Inverse constant-elasticity demand

consumption. Thus, the total revenue function measures how much consumers spend at every given market price, and not necessarily the revenue collected by producers.

The *marginal-revenue* function (again, more appropriately termed the "marginal expenditure") shows the amount by which total revenue increases when the consumers slightly increase the amount they buy. Formally we define the marginal-revenue function by $MR(Q) \equiv \frac{\mathrm{d}TR(Q)}{\mathrm{d}Q}$.

For the linear demand case we can state the following:

Proposition 1.1 *If the demand function is linear, then the marginal-revenue function is also linear, has the same intercept as the demand, but has twice the (negative) slope. Formally,* $MR(Q) = a - 2bQ$.

Proof.

$$MR(Q) = \frac{\mathrm{d}TR(Q)}{\mathrm{d}Q} = \frac{\mathrm{d}(aQ - bQ^2)}{\mathrm{d}Q} = a - 2bQ.$$

■

For the constant-elasticity demand we do not draw the corresponding marginal-revenue function. However, we consider one special case where $\epsilon = 1$. In this case, $p = aQ^{-1}$, and $TR(Q) = a$, which is a constant. Hence, $MR(Q) = 0$.

You have probably already noticed that the demand elasticity and the marginal-revenue functions are related. That is, Figure 1.2 shows that $MR(Q) = 0$ when $\eta_p(Q) = 1$, and $MR(Q) > 0$ when $|\eta_p(Q)| > 1$. The complete relationship is given in the following proposition.

Proposition 1.2

$$MR(Q) = p(Q)\left[1 + \frac{1}{\eta_p(Q)}\right].$$

Proof.

$$\begin{aligned} MR(Q) &\equiv \frac{\mathrm{d}TR(Q)}{\mathrm{d}Q} = \frac{\mathrm{d}[p(Q)Q]}{\mathrm{d}Q} = p + Q\frac{\partial p(Q)}{\partial Q} \\ &= p\left[1 + \frac{Q}{p}\frac{1}{\frac{\partial Q(p)}{\partial p}}\right] = p\left[1 + \frac{1}{\eta_p(Q)}\right]. \end{aligned}$$

■

1.7.1 Cross Elasticity & Income elasticity

$$\eta_{x,p_x} \stackrel{\text{def}}{=} \frac{\partial x}{\partial p_y}\frac{p_y}{x}$$

$$\eta_{x,I} \stackrel{\text{def}}{=} \frac{\partial x}{\partial I}\frac{I}{x}$$

1.7.2 Characteristics and relationships among the various elasticity functions

Let,

$$\lambda_x \stackrel{\text{def}}{=} \frac{p_x x}{I}, \quad \lambda_y \stackrel{\text{def}}{=} \frac{p_y y}{I}$$

be the fraction of the consumer's income spent on goods x and y, respectively.

1. The sum of all elasticities equals zero.

$$\eta_{x,p_x} + \eta_{x,p_y} + \eta_{x,I} = 0$$

2. Weighted sum of self- and cross elasticities equals minus fraction of income spend on x:

$$\lambda_x \eta_{x,p_x} + \lambda_y \eta_{y,p_x} = -\lambda_x$$

3. Sum of all income elasticities equals 1:

$$\lambda_x \eta_{x,I} + \lambda_y \eta_{y,I} = 1$$

Verification of the above identities: Use the Cobb-Douglas demand functions, subsection 1.6.3, to demonstrate.

1.7.3 Hicks decomposition: substitution and income effects

- Draw
  1. ICC (Income-consumption curve): set of all bundles at given prices while varying income only. Define:
     (a) Normal goods
     (b) Inferior goods
  2. PCC (Price-consumption curves): set of all bundles at a given income varying p_x only
- Draw three graphs showing Hicks decomposition for normal, inferior, and Giffen good x
- Demonstrate that monotonicity implies that at least one good must be normal
- Formalize this decomposition using the *Slutski equation*:

$$\frac{\partial x}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U_0} - x \left.\frac{\partial x}{\partial I}\right|_{p_x/p_y}$$

  Remark: The consumption level of x determines the impact of the income effect. For example, if $x = 0$, there is no income effect.
  Remark: Demonstrate normal, inferior, and Giffen (using $\pm$)

1.8 Compensated demand

- **Purpose:** to draw the demand function assuming a given welfare level (i.e., allowing only substitution effects, thereby adjusting income to keep the consumer on the same utility level U_0.
- Derivation:

$$\min_{x,y} p_x x + p_y y \quad \textbf{s.t. } U(x,y) - U_0 = 0$$

  2 equations (the tendency conditions plus the constraint)
- **Example:** $U(x,y) = xy$ yielding

$$\frac{p_x}{p_y} = \frac{MU_x}{MU_y} = \frac{y}{x} \quad \& \quad xy = U_0 \quad \Longrightarrow \quad x = \sqrt{\frac{U_0 p_y}{p_x}} \quad \& \quad y = \sqrt{\frac{U_0 p_x}{p_y}}$$

- **Ordinary versus compensated demand:** Figure 1.4 illustrates the relative steepness for the case where x is normal and x is inferior.
- **Example:** $U = xy$, $I = 12$, $p_x = 1$, $p_y = 2$ (hence, $x = 6$, $y = 3$ and $U_0 = 18$). In this case (x is normal),

$$\frac{\partial x}{\partial p_x} = -\frac{I}{2(p_x)^2} = -6$$

  However,

$$\left.\frac{\partial x}{\partial p_x}\right|_{U_0} = -\frac{\sqrt{U_0 p_y}}{2\sqrt{(p_x)^3}} = -3$$

  Remark: the slopes of the inverse demand functions are $-1/6$ and $-1/3$, respectively.

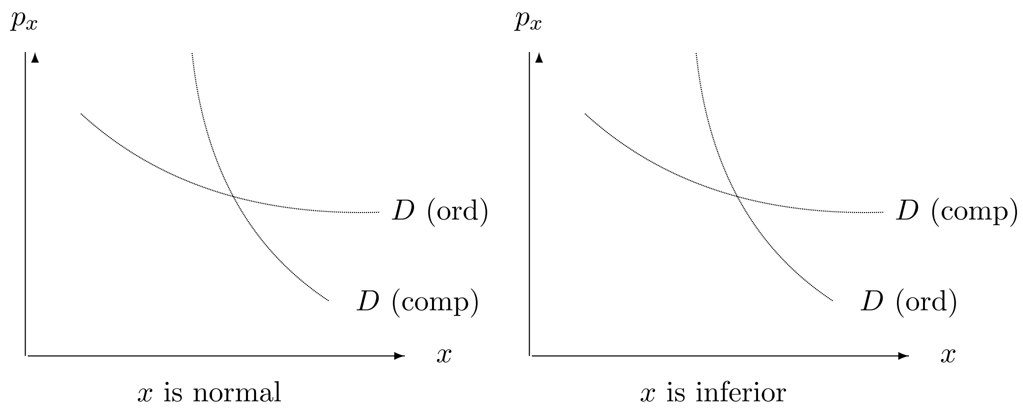

Figure 1.4: *Normal:* Ord is more elastic (subs. and income effect same direction. *Inferior:* Ord is less elastic

1.9 Slutski decomposition

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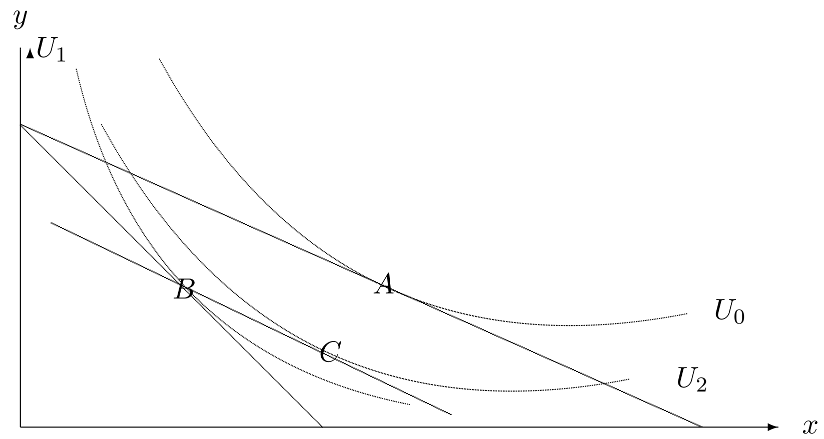

Figure 1.5: Slutski decomposition: Given a fixed real income, welfare rises

1.10 Revealed preferences

- **Goal:** To determine changes in consumer welfare without knowing utility functions. We rely only on one or two axioms concerning consumer behavior
- **Notation:** Let $\vec{x}^t \stackrel{\text{def}}{=} (x_1^t, \ldots, x_N^t)$ be the bundled purchased at prices $\vec{p}^t \stackrel{\text{def}}{=} (p_1^t, \ldots, p_N^t)$, $t = 0, 1$. *Note:* $t = 0$ will be called the base period.

- **Definition:** A bundle $\vec{x}^0$ is *revealed preferred* to bundle $\vec{x}^1$ if $\vec{x}^0$ was purchased while $\vec{x}^1$ was also affordable. Formally, if

$$\sum_{i=1}^{N} p_i^0 x_i^1 \leq \sum_{i=1}^{N} p_i^0 x_i^0$$

- **Weak Axiom of Revealed Preference (WARP):** If a bundle $\vec{x}^0$ is revealed preferred to $\vec{x}^1$ then $\vec{x}^1$ must never be revealed preferred to $\vec{x}^0$. Formally,

$$\sum_{i=1}^{N} p_i^0 x_i^1 \leq \sum_{i=1}^{N} p_i^0 x_i^0 \quad \Longrightarrow \quad \sum_{i=1}^{N} p_i^1 x_i^0 > \sum_{i=1}^{N} p_i^1 x_i^1$$

- **Examples:**

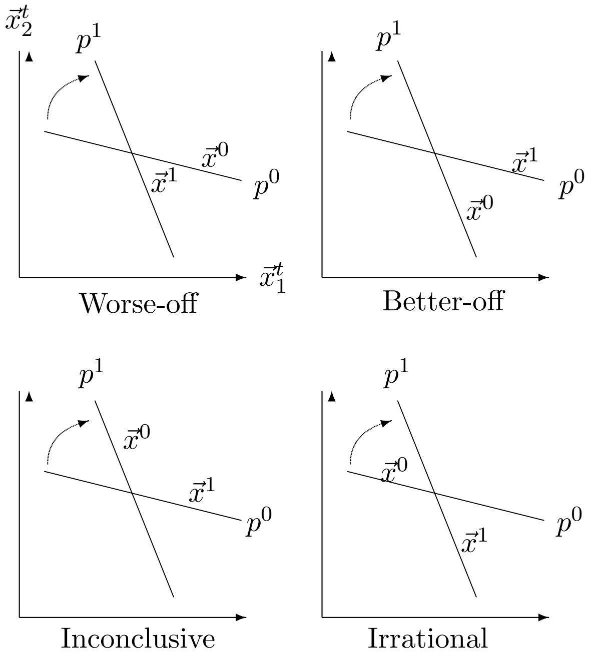

Figure 1.6: Using revealed preference

1.11 Indexes

- **Goals:** Define indexes which
  1. would indicate changes in consumer welfare
  2. extend the graphical analysis to $N > 2$ goods.
- Laspeyres Quantity Index (base prices)

$$Q_L \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{N} p_i^0 x_i^1}{\sum_{i=1}^{N} p_i^0 x_i^0} = \frac{\sum_{i=1}^{N} p_i^0 x_i^1}{I^0}.$$

- Paasche Quantity Index (After-change prices)

$$Q_P \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{N} p_i^1 x_i^1}{\sum_{i=1}^{N} p_i^1 x_i^0} = \frac{I^1}{\sum_{i=1}^{N} p_i^1 x_i^0}.$$

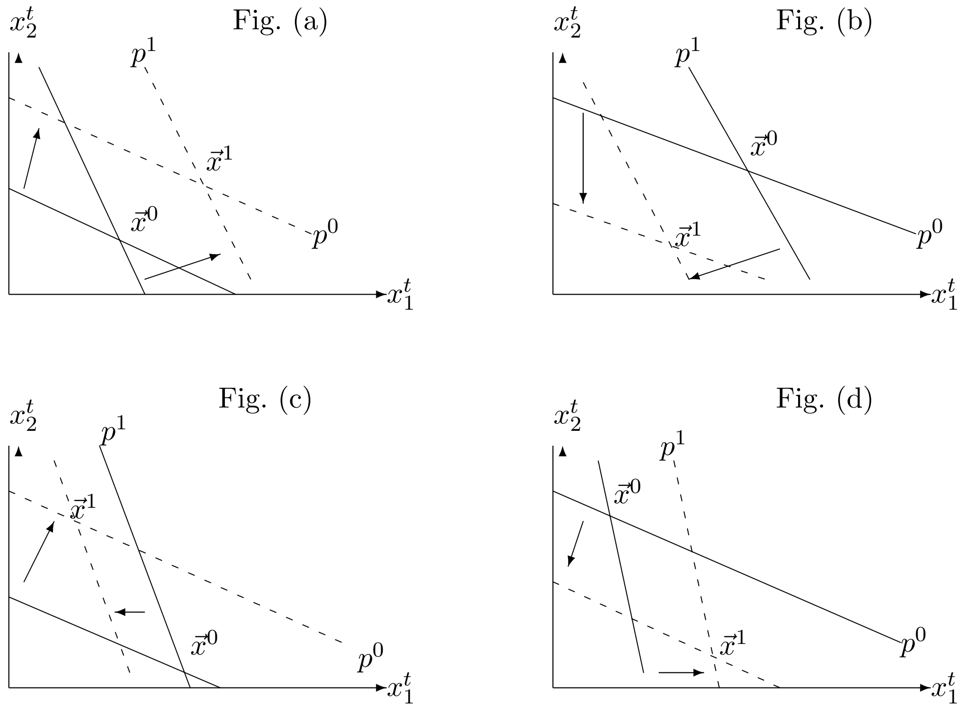

Figure 1.7: Welfare consequences of quantity indexes

Figure 1.7 should be interpreted as follows:

(a) $Q_L > 1$ and $Q_P > 1$ implying consumers are better off.

(b) $Q_L < 1$ and $Q_P < 1$ implying consumers are worse off.

(c) $Q_L > 1$ and $Q_P < 1$ implying inconclusive (depending on indifference curves).

(d) $Q_L < 1$ and $Q_P > 1$ implying inconsistency (a change in tastes).

- Laspeyres Price Index (base-period basket):

$$P_L \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{N} x_i^0 p_i^1}{\sum_{i=1}^{N} x_i^0 p_i^0} = \frac{\sum_{i=1}^{N} x_i^0 p_i^1}{I^0}.$$

- Paasche Price Index (later-period basket):

$$P_P \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{N} x_i^1 p_i^1}{\sum_{i=1}^{N} x_i^1 p_i^0} = \frac{I^1}{\sum_{i=1}^{N} x_i^1 p_i^0}.$$

- Price-Rise Compensation (To-sefet Yoker):
  Assume no change in income and look for compensation according to Slutski. Thus,

$$\Delta I = \sum_{i=1}^{N} p_i^1 x_i^0 - \sum_{i=1}^{N} p_i^0 x_i^0.$$

  Thus, the compensation as percentage of the base income is

$$\frac{\Delta I}{I^0} = \frac{\sum_{i=1}^{N} p_i^1 x_i^0 - \sum_{i=1}^{N} p_i^0 x_i^0}{\sum_{i=1}^{N} p_i^0 x_i^0} = P_L - 1.$$

1.12 Consumer surplus

First, distinguish between *gross* and *net* consumer surplus. In what follows we discuss a common procedure used to approximate consumers' gain from buying by focusing the analysis on linear demand functions. Figure 1.8 illustrates how to calculate the consumer surplus, assuming that the market price is p.

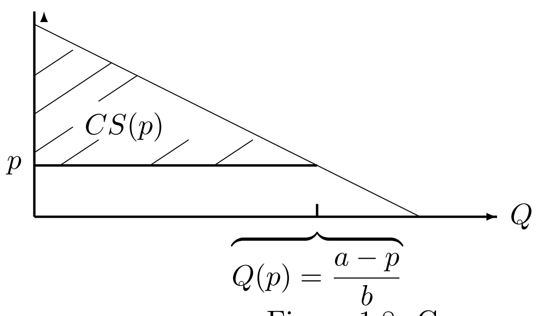

Figure 1.8: Consumers' surplus

For the $p = a - Q$ case, e

$$CS(p) \equiv \frac{(a-p)Q(p)}{2} = \frac{(a-p)^2}{2b}. \quad (1.2)$$

Note that $CS(p)$ must always increase when the market price is reduced, reflecting the fact that consumers' welfare increases when the market price falls.

1.12.1 consumer Surplus: Quasi-Linear Utility

We demonstrate that when consumer preferences are characterized by a class of utility functions called *quasi-linear utility function*, the measure of consumer surplus equals exactly the total utility consumers gain from buying in the market.

Consider a consumer who has preferences for two items: money (m) and the consumption level (Q) of a certain product, which he can buy at a price of p per unit. Specifically, let the consumer's utility function be given by

$$U(Q, m) \equiv \sqrt{Q} + m. \quad (1.3)$$

Now, suppose that the consumer is endowed with a fixed income of I to be spent on the product or to be kept by the consumer. Then, if the consumer buys Q units of this product, he spends pQ on the product and retains an amount of money equals to $m = I - pQ$. Substituting into (1.3), our consumer wishes to choose a product-consumption level Q to maximize

$$\max_{Q} U(Q, I - pQ) = \sqrt{Q} + I - pQ. \tag{1.4}$$

The first-order condition is given by $0 = \partial U/\partial Q = 1/(2\sqrt{Q}) - p$, and the second order by $\partial^2 U/\partial Q^2 = -1/(4Q^{-3/2}) < 0$, which constitutes a sufficient condition for a maximum.

The first-order condition for a quasi-linear utility maximization yields the inverse demand function derived from this utility function, which is given by

$$p(Q) = \frac{1}{2\sqrt{Q}} = \frac{Q^{-1/2}}{2}. \tag{1.5}$$

Thus, the demand derived from a quasi-linear utility function is a constant elasticity demand function, illustrated earlier in Figure 1.3, and is also drawn in Figure 1.9.

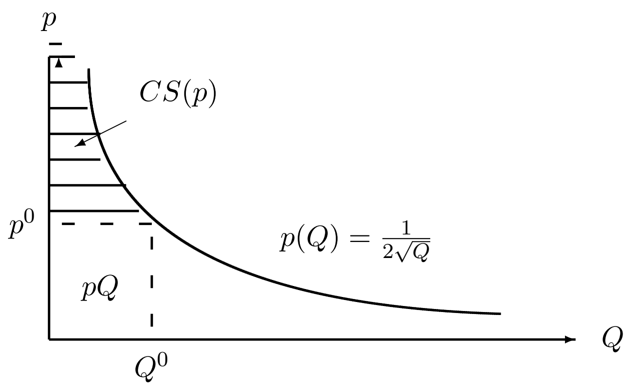

Figure 1.9: Inverse demand generated from a quasi-linear utility function

The shaded area in Figure 1.9 corresponds to what we call consumer surplus. The purpose of this appendix is to demonstrate the following proposition.

Proposition 1.3 *If a demand function is generated from a quasi-linear utility function, then the area marked by $CS(p)$ in Figure 1.9 measures exactly the utility the consumer gains from consuming Q^0 units of the product at a market price p^0.*

Proof. The area $CS(p)$ in Figure 1.9 is calculated by

$$\begin{aligned} CS(p) &\equiv \int_0^{Q^0} \left(\frac{1}{2\sqrt{Q}}\right) dQ - p^0 Q^0 \\ &= \sqrt{Q^0} - p^0 Q^0 = U(Q^0, I - p^0) + \text{ constant}. \end{aligned} \tag{1.6}$$

■

1.12.2 Equivalent and Compensation variations

- Purpose: How to compensate consumers when prices hike (say, resulting from an imposition of a tax on good X)
- Problem: To compensate for a utility loss resulting from a price hike we need to know consumers' indifference curves
- Propose 2 measures (assuming that we know the indifferent curves):

  Equivalent variation (EV): How much the consumer would be *willing to pay* to avoid the tax [using 'old' prices (before the change)]

  Compensating variation (CV): How much we need to *compensate* the consumer for imposing the tax [using 'new' prices (after the change)]

- Result: Either EV $\geq$ CS $\geq$ CV or EV $\leq$ CS $\leq$ CV
- Compensation in terms of Y (or assume $p_y = 1$), Figure 1.10 illustrates the compensation levels

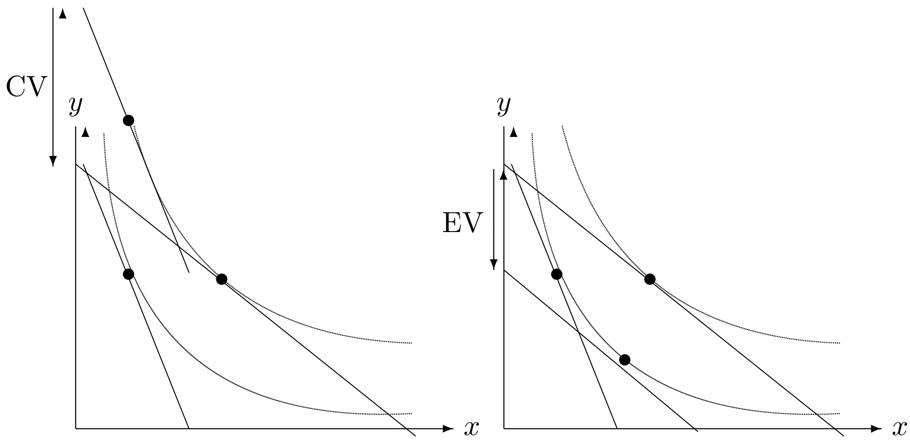

Figure 1.10: *Left:* Compensating variation (new prices). *Right:* Equivalent variation (old prices).

- Quasi-linear utility functions:

1.13 Commodity Endowment

- A consumer is endowed with x_0 units of X and y_0 units Y
- **Interpretation:** Fruits & vegetables grown in your own garden, time, and Labor
- Let M be monetary income, then for *given* p_x and p_y,

$$I = M + p_x x_0 + p_y y_0$$

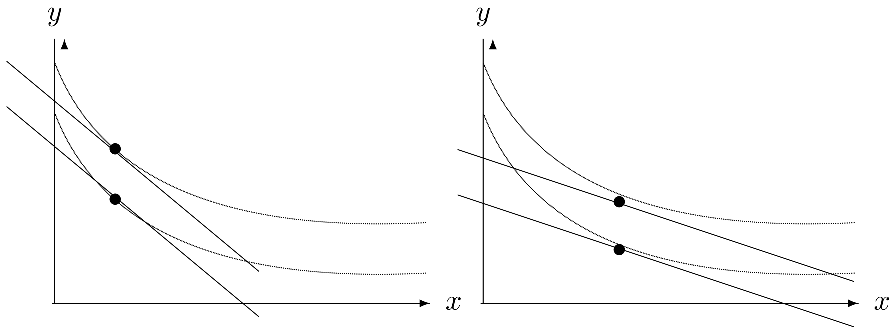

Figure 1.11: Quasi-linear utility case: When $p_y = 1$, EV=CV=CS since the difference between IC is independent of the slope p_x/p_y

so income is decomposed into monetary and real incomes

- **Proposition:** Let $M = 0$ (non-monetary income), then the budget varies only if there is a change in p_x/p_y
- If $M = 0$, draw
$$y = y_0 + \frac{p_x}{p_y} x_0 - \frac{p_x}{p_y} x$$
- Show how to draw the budget constraint for $M > 0$. *Hint:* add M/p_x and M/p_y to the relevant intercepts
- **Assumption:** $M = 0$ unless otherwise specified!
- **Analyze:** welfare effects of changing p_x/p_y for 2 cases
  1. Endowment in X only: $x_0 > 0$ and $y_0 = 0$, in which case
  $$\left.\frac{p_x}{p_y}\right\uparrow \quad \text{iff} \quad U \downarrow$$
  2. Endowment in 2 goods: $x_0 > 0$ and $y_0 > 0$, in which case a change in p_x/p_y causes a "rotation" of the budget constraint, so welfare analysis is more complicated (DO IT using GRAPHS!)

Excess demand function Solve

$$\max_{x,y} U(x, y) \quad \textbf{s.t.} \quad p_x x + p_y y \le p_x x_0 + p_y y_0$$

then, the excess demand function for X is defined by

$$x^{\text{ED}}\left(\frac{p_x}{p_y}, x_0, y_0\right) \stackrel{\text{def}}{=} x(p_x, p_y, I) - x_0, \quad \text{where } I \stackrel{\text{def}}{=} p_x x_0 + p_y y_0$$

Example: Cobb Douglas:

$$x^{\text{ED}} = x - x_0 = \frac{I}{2p_x} - x_0 = \frac{p_y y_0 - p_x x_0}{2p_x} \quad = 0 \text{ when } \frac{p_x}{p_y} = \frac{y_0}{x_0}$$

Figure 1.12 plots it on the p_x/p_y—$(x - x_0)$ space.

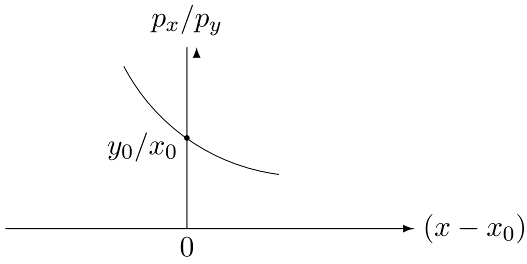

Figure 1.12: Excess demand function: Cobb-Douglas case

Interpretation of the ED curve: (1) International trade: how much a country imports (exports, if negative) of a good as a function of the world price ratio. If X is imported, then p_x/p_y is called the *terms of trade.*
(2) If x_0 is current income, $x - x_0$ is excess consumption over current income, which is called *savings.*

1.14 Labor supply

Each individual is endowed with 24 hours to be allocated between *leisure* and *work.* Major issues:

1. At a given wage rate, w, how many hours of works will be supplied by the individual?
2. How would a change in w affect the amount of labor supplied
3. How would an income tax affect labor supply?

The Model Two goods: Leisure (ℓ), priced by w, (wage rate) and other goods (y), whose price is $p_y = p$.

Budget Constraint Figure 1.13 illustrates various budget constraints

Labor supply

- Draw equilibria for varying real wage, w/p and show upward- and backward-bending labor supply curves.
- For overtime wage, $w_1 > w_0$ show the possibility of multiple equilibria

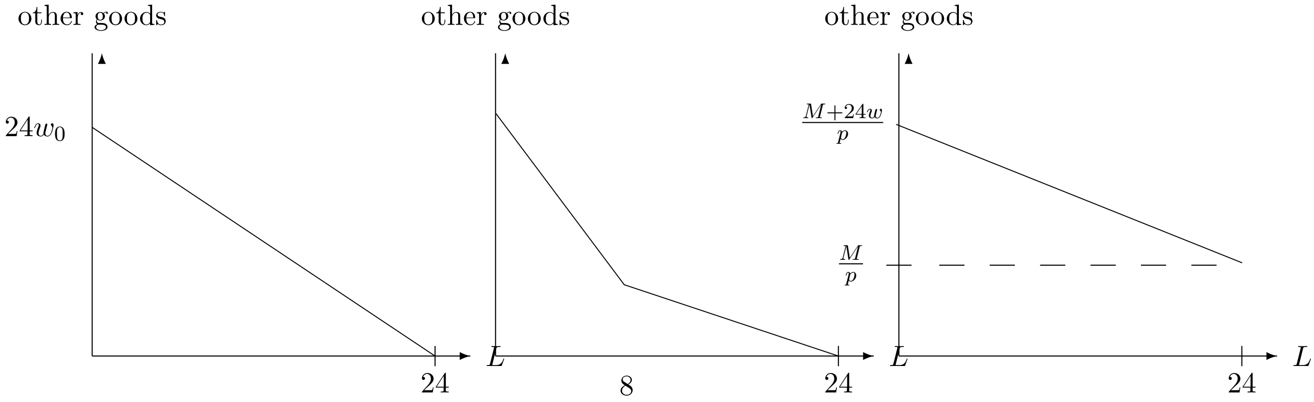

Figure 1.13: *Left:* uniform wage. *Middle:* overtime wage beyond 8 hours. *Right:* Non-labor income

- Cobb-Douglas example: Special case, since labor supply is given if there is only labor income,

$$\ell(w,p) = \frac{24w}{2w} = 12, \quad \text{and} \quad y(w,p) = \frac{24w}{2p}$$

- Overtime wage: Take $w_1 = 1$, $w_2 = 2$ for $\ell \leq 16$ (8 hours of work), $p = 1$, and show that $\ell = 10$, $24 - \ell = 14$ (overtime of 4 hours), and $y = 20$.

1.15 Saving & Loans

- Intertemporal consumers, live more than one period (2 for our purposes: present and future).
- Two goods: consumption now, c_1, and consumption in the future, c_2
- Assumption: no inflation, $p_1 = p_2 \stackrel{\text{def}}{=} 1$
- Hence, real interest rate, r equals i (nominal interest rate)
- I_1 income at present, I_2 future income
- Present and future values are

$$PV = I_1 + \frac{I_2}{1+\rho}, \quad FV = (1+r)I_1 + I_2$$

- Budget Constraint:

$$c_1 = I_1 + \frac{I_2 - c_2}{1+r}, \quad \text{or} \quad c_1 + \frac{c_2}{1+r} = I_1 + \frac{I_2}{1+r}$$

- Slope: $1/(1+r)$
- Utility function: $U(c_1, c_2)$ (monotonic, etc.)

- Consumer equilibrium:
$$\frac{MU_1}{MU_2} = 1 + r$$

- Definition: A consumer is said to have

  preference for the present if, $U(a, b) > U(b, a)$ iff $a > b$

  preference for the future if, $U(a, b) > U(b, a)$ iff $a < b$

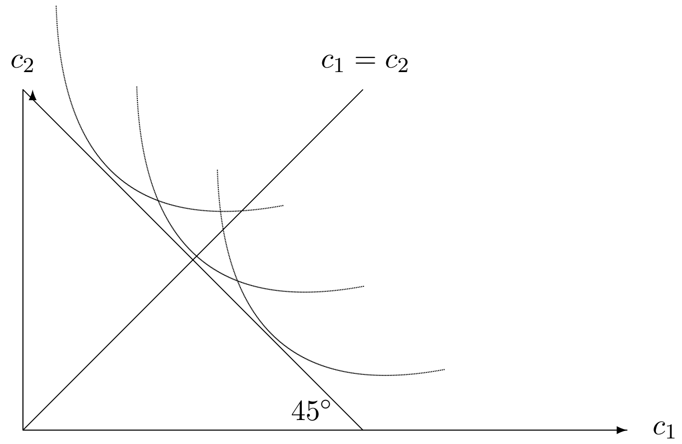

Figure 1.14: *Right:* Present preference, *Left:* Preference for the Future

- Income changes (parallel shifts) (increase in consumption (saver can turn into a borrower, and vice versa

- Interest-rate increase, 2 cases:

  Borrower: Assuming normality, present consumption falls (may even become a lender) since substitution and income effects are of the same signs

  Lender: Assuming normality, both income- and substitution have different signs. Saving may go up or down. However, the consumer will not turn into a borrower!

- Imperfect market: Lender earns r_1, borrower pays r_2 where $r_2 > r_1$ (hence, a kink at the endowment point).

- **Example with Imperfect Capital Market:** Let, $I_1 = 100$, $I_2 = 110$, $r_b = 10\%$, $r_\ell = 5\%$, and $U = c_1 c_2$.
$$\text{PV} = 100 + \frac{110}{1.1} = 200, \quad \text{FV} = 105 + 110 = 215$$

  If *lender* solve,
$$\frac{c_2}{c_1} = 1.05 \quad c_2 = 215 - 1.05c_1 \Longrightarrow c_1 \approx\ 102 > 100 \quad \text{borrower, a contradiction}$$

If *borrower*,

$$\frac{c_2}{c_1} = 1.10 \quad c_2 = 220 - 1.1c_1 \Longrightarrow c_1 = 100 \quad \text{no borrowing no lending}$$

- **Inflation:** Let i be the nominal interest rate, and π the inflation rate.

$$\text{PV} = \frac{I_1 + \frac{I_2}{1+i}}{p_1} \quad \text{FV} = \frac{(1+i)I_1 + I_2}{p_2} \Longrightarrow \text{slope} = \frac{\text{FV}}{\text{PV}} = \frac{p_1}{p_2}(1+i)$$

Define $1 + \pi \overset{\text{def}}{=} p_2/p_1$. Then, define the *real* interest rate, r by

$$1 + r \overset{\text{def}}{=} \frac{1+i}{p_1/p_2} = \frac{1+r}{1+\pi}.$$

Then,

$$r = \frac{1+r}{1+\pi} - 1 = \frac{r-\pi}{1+\pi} \approx r - \pi.$$

Note: Price changes do change the endowment point in the c_2–c_1 space.

- **Bonds:** with maturity at period $t = T$, paying x/period with face value $\$F$:

$$PV = \frac{x}{1+i} + \frac{x}{(1+i)^2} + \frac{x}{(1+i)^3} + \cdots + \frac{F+x}{(1+i)^T}.$$

A consol:

$$PV = \sum_{t=0}^{\infty} \frac{x}{(1+i)^t} = \frac{x}{i}.$$

Lecture 2
Uncertainty Models

- income may be not be perfectly foreseen (uncertain income). e.g.,
  1. Business income fluctuations (fluctuating demand)
  2. Nature's moves (fire, earthquakes, etc.)
- N states of nature indexed by i. e.g., rain or shine ($N = 2$)
- Event$_i$ = a certain income (loss) in state of nature i
- π_i = probably for each event

2.1 The Expected Utility Hypothesis

- Assumption: the utility is the expected value of utilities under each state of nature
- Formally, let $u(c_i)$ be the utility of consumption in one state i, $i = 1, 2$, then

$$U(c_1, c_2) \stackrel{\text{def}}{=} \pi_1 u(c_1) + \pi_2 u(c_2)$$

- Remark: Monotonicity invariance is no longer satisfied since

$$U^2 = [\pi_1 v(c_1) + \pi_2 v(c_2)]^2$$

  is no longer an expected utility
- Hence, *affine transformation* is the only transformation that maintains expected utility: $F(U) \stackrel{\text{def}}{=} aU + b$, $a > 0$.

2.2 Defining risk aversion

- Depends on the concavity of the function $u(c_i)$
- Suppose w_1 with π_1; and w_2 with π_2
- Expected consumption: $Ew = \pi_1 \times w_1 + \pi_2 \times w_2$
- Expected utility $Eu(w) = \pi_1 \times u(w_1) + \pi_2 \times u(w_2)$
- Risk aversion if: $u(Ew) > Eu(w)$
- Risk lover if: $u(Ew) > Eu(w)$
- Risk neutral if: $u(Ew) = Eu(w)$

- Show graphs
- **Examples:** Let $\pi_1 = \pi_2 = 1/2$, and $w_1 = 1$, $w_2 = 9$, and
  1. $U = \pi_1\sqrt{w_1} + \pi_2\sqrt{w_2}$
  2. $V = \pi_1(w_1)^2 + \pi_2(w_2)^2$

Probability:	0.5	0.5
State of the World:	Bad	Good
Event (r.v.):	$w_g = 1$	$w_b = 9$
$u(w_i) =$	1	3
$v(w_i) =$	1	81

Table 2.1: Expected utility

$$
\begin{aligned}
Ew &= 0.5 \times 1 + 0.5 \times 9 = 5 \\
u(Ew) &= \sqrt{5} = 2.24 \\
Eu(w) &= 0.5 \times 1 + 0.5 \times 3 = 2 \\
v(Ew) &= 25 \\
Ev(w) &= 0.5 \times 1 + 0.5 \times 81 = 41
\end{aligned} \tag{2.1}
$$

2.3 Insurance

- Purpose: to change probabilities of event for the insured
- 2 states of nature called: Good (probability $\pi_g = 0.99$) and bad (probability $\pi_b = 0.01$)
- $W = \$35,000$. Loss of $\$10,000$ (theft) can occur with probability $\pi = 0.01$ (1 percent)
- Draw the contingent bundle in the *contingent consumption plan* space ($c_g - c_b$):
- γ= insurance premium = price of every \$1 of insurance (to be paid by the insurance company in the bad state only!)
- K ($K \leq W$)= amount of insurance purchased
- Revised contingent consumption with $\$K$ of insurance: $c_g = \$35,000 - \gamma K$ and $c_b = \$35,000 - \gamma K - \$10,000 + \$K$
- Budget constraint with the availability of insurance:

$$\text{slope} = \frac{\Delta c_g}{\Delta c_b} = \frac{35,000 - (35,000 - \gamma K)}{25,000 - (25,000 - \gamma K + K)} = -\frac{\gamma}{1-\gamma}$$

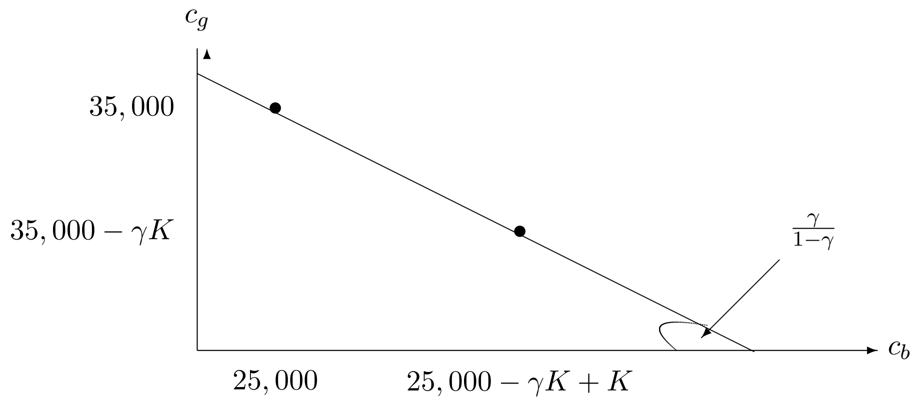

Figure 2.1: Contingent consumption with and without insurance

2.4 Consumer equilibrium: optimal insurance level

Let $U = \pi_g \ln c_g + \pi_b \ln c_b$.

$$MRS_{c_g,c_b} = \frac{\pi_b/c_b}{\pi_g/c_g} = \frac{\pi_b}{\pi_g}\frac{c_g}{c_b}$$

Since $c_g = \$35,000 - \gamma K$ and $c_b = \$25,000 - \gamma K + K$,

$$MRS_{c_g,c_b} = \frac{\pi_b}{\pi_g}\frac{35,000 - \gamma K}{25,000 - \gamma K + K}$$

Hence

$$MRS_{c_g,c_b} = \frac{\gamma}{1-\gamma} \Longrightarrow \frac{\pi_b}{\pi_g}\frac{35,000 - \gamma K}{25,000 - \gamma K + K} = \frac{\gamma}{1-\gamma},$$

from which we can solve for K (not always easy)!

2.5 Free entry competitive insurance industry

- Free entry into an industry occurs as long as firms make above normal profit
- Look for profit of a representative firm and equate it to zero
- profit= $\gamma K - \pi_b K$
- profit declines to zero when γ declines to π_b
- $\gamma = \pi_b$ is called a *fair premium*
- substituting into the consumer equilibrium,

$$\frac{0.01}{0.99}\frac{35,000 - 0.01K}{25,000 - 0.01K + K} = \frac{0.01}{1 - 0.01}$$

  implies that $K = 10,000$ Therefore,
  Proposition: A risk averse consumer facing a "fair" premium will always *fully insure*!

2.6 Diversification of portfolios to reduce risk

Should one invest in one or two risky assets? Hence, any risk-averse investor will choose to diversify.

Event	π_i	Project 1 (profit per 1$)	Project 2:	50¢ in each
Rain:	1/2	5	10	7.5
Shine:	1/2	10	5	7.5
Expected profit:		7.5	7.5	7.5

Table 2.2: Gain from diversifying risk (splitting $1 between 2 projects, secures a profit of $15)

Lecture 3
Producer Theory

3.1 The Production Function

- Input space: k and ℓ, $(\ell, k) \in \mathbb{R}_+$
- Output: Y (y units of Y)
- Production function: $F : \mathbb{R}^2_+ \to \mathbb{R}_+$, $y = f(\ell, k)$
- Isoquant: set of all factor bundles (ℓ, k) satisfying $f(\ell,) = y_0$
- Marginal product:

$$\text{MP}_\ell \stackrel{\text{def}}{=} \frac{\partial f(\ell, k)}{\partial \ell}, \quad \text{MP}_k \stackrel{\text{def}}{=} \frac{\partial f(\ell, k)}{\partial k}$$

- Average product:

$$AP_\ell(\ell, k) \stackrel{\text{def}}{=} \frac{y}{\ell}, \quad AP_k(\ell, k) \stackrel{\text{def}}{=} \frac{y}{k}$$

- Rate of Technical Substitution:

$$RTS_{k,\ell} = \left.\frac{\partial k}{\partial \ell}\right|_{y=y_0} = \frac{\text{MP}_\ell}{\text{MP}_k}$$

- Homogeneous Degree α: $f(\lambda\ell, \lambda k) = \lambda^\alpha f(\ell, k)$
- Returns to Scale: Let $\lambda > 1$,

  DRS: if $f(\lambda\ell, \lambda k) < \lambda f(\ell, k)$
  IRS: if $f(\lambda\ell, \lambda k) > \lambda f(\ell, k)$
  CRS: if $f(\lambda\ell, \lambda k) = \lambda f(\ell, k)$

- Derive MP_i, AP_i, draw indifference curves, and find returns to scale for

  Cobb-Douglas: $y = \ell^\alpha k^\beta, \alpha, \beta > 0$
  Perfect Substitutes: $y = \alpha\ell + \beta k$
  Perfect Complements: $y = \min\{\alpha\ell, \beta k\}$
  CES: $y = (\ell^\alpha + k^\alpha)^\beta$
  Quasi-Linear: $y = \sqrt{\ell} + k$

3.2 Properties of CRS production functions

MP depends only on k/ℓ: CRS $\Longrightarrow$ $\forall\lambda > 0, \lambda f(\lambda\ell, \lambda k) = \lambda f(\ell, k)$. Differentiation both sides with respect to ℓ yields, $\lambda f_\ell(\lambda\ell, \lambda k) = \lambda f_\ell(\ell, k)$, or $f_\ell(\lambda\ell, \lambda k) = f_\ell(\ell, k)$. Define $\lambda \stackrel{\text{def}}{=} 1/\ell$ yields

$$f_\ell\left(1, \frac{k}{\ell}\right) = f_\ell(\ell, k).$$

Let $y_1 = f(\ell_1, k_1)$, $y_2 = f(\ell_2, k_2)$, and $y_3 = f(\ell_3, k_3)$. Then,

$$\frac{\ell_2}{\ell_1} = \frac{\ell_3}{\ell_2} = \frac{k_2}{k_1} = \frac{k_3}{k_2} \stackrel{\text{def}}{=} \lambda \Longrightarrow \frac{y_2}{y_1} = \frac{y_3}{y_2} = \lambda$$

That is, distances measured on a ray from the origin are proportional to the output represented on the corresponding indifference curves.
Proof. First, note that

$$\frac{\ell_2}{\ell_1} = \frac{k_2}{k_1} \Longrightarrow \frac{k_1}{\ell_1} = \frac{k_2}{\ell_2} \text{ and } \frac{\ell_3}{\ell_2} = \frac{k_3}{k_2} \Longrightarrow \frac{k_3}{\ell_3} = \frac{k_2}{\ell_2}$$

Then,

$$\frac{y_2}{y_1} = \frac{f(\ell_2, k_2)}{f(\ell_1, k_1)} = \frac{\ell_2 f\left(1, \frac{k_2}{\ell_2}\right)}{\ell_1 f\left(1, \frac{k_1}{\ell_1}\right)} = \frac{\ell_2}{\ell_1} = \lambda$$

Similarly,

$$\frac{y_3}{y_2} = \frac{f(\ell_3, k_3)}{f(\ell_2, k_2)} = \frac{\ell_3 f\left(1, \frac{k_3}{\ell_3}\right)}{\ell_2 f\left(1, \frac{k_2}{\ell_2}\right)} = \frac{\ell_3}{\ell_2} = \lambda$$

■

Average product depends only on k/ℓ:
Proof.

$$\frac{f(\ell, k)}{\ell} = \frac{\ell f\left(1, \frac{k}{\ell}\right)}{\ell} = f\left(1, \frac{k}{\ell}\right)$$

Euler's Theorem: CRS $\Longrightarrow$ $y = f(\ell, k) = \ell \text{MP}_\ell(\ell, k) + k\text{MP}_k$ i.e., total output is the sum of each input weighted by its MP
Proof. CRS $\Longrightarrow$ $\lambda f(\ell, k) = f(\lambda\ell, \lambda k)$. Differentiating both sides with respect to λ,

$$f(\ell, k) = \ell f_\ell(\lambda\ell, \lambda k) + k f_k(\lambda\ell, \lambda k) = \ell f_\ell(\ell, k) + k f_k(\ell, k)$$

Dividing by y yields

$$1 = \frac{\partial y}{\partial \ell}\frac{\ell}{y} + \frac{\partial y}{\partial k}\frac{k}{y} = \eta_{y,\ell} + \eta_{y,k}$$

i.e., the sum of output elasticities equals to 1

3.3 Short-run Production Function

- Definition: short-run is the time period in which the firm cannot alter the amount of rented capital. I.e,, $k = k_0$
- Examples: land, office space
- Draw TP(ℓ, k_0) (first increasing MP, then decreasing)
- Draw below the *corresponding* AP(ℓ, k_0) and MP(ℓ, k_0)
- Proposition: At $\ell^{\max}$ which minimizes AP(ℓ, k_0), AP$(\ell^{\max}, k_0) =$ MP$(\ell^{\max}, k_0)$
  Proof.
  $$0 = \frac{dAP}{d\ell} = \frac{d\left(\frac{TP}{\ell}\right)}{d\ell} = \frac{MP_\ell - TP}{\ell^2}$$
  ■

3.4 Cost Functions

- Let, W wage rate, R rental on capital
- $TC(W, R, y)$ maps factor-rental prices to \$s
- Emphasize duality: cost function can derived from a production function, and vise versa.
- Marginal Cost: MC$(y) \stackrel{\text{def}}{=} \partial$TC$(y)/\partial y$
- Average Cost: AC$(y) \stackrel{\text{def}}{=}$ TC$(y)/y$

3.4.1 Single-factor case: A demonstration

How to derive the cost function from a production function $y = \ell^\gamma$? Let, W wage rate and $\gamma > 0$.

1. Input-requirement function: $\ell = y^{1/\gamma}$
2. cost means payment to factors: $TC(W, y) = W\ell = Wy^{1/\gamma}$.
3. Note: return to scale (see Figure 3.1):
   $$(\lambda\ell)^\gamma > \lambda\ell^\gamma \quad \text{iff} \quad \gamma > 1$$

3.4.2 Cost minimization and long-run cost functions

- Given W and R, find ℓ and k that minimize cost of producing y_0 units of output.
  $$\min_{\ell,k} W\ell + Rk \quad \textbf{s.t.} \quad f(\ell, k) \geq y_0$$

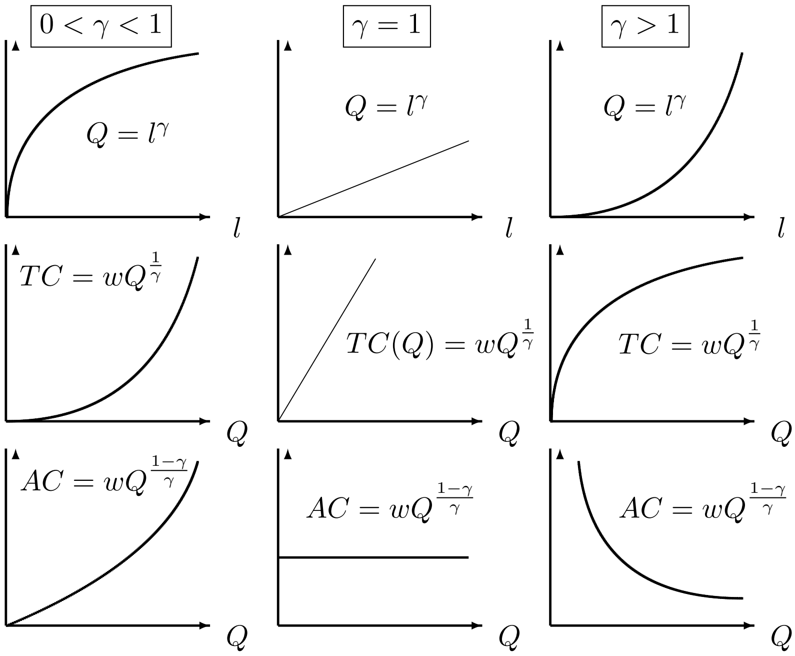

Figure 3.1: Duality between cost- and production functions

- Discuss corner vs. interior solutions. If $\ell^{\min}$, $k^{\min} > 0$,

$$\frac{\text{MP}_\ell}{\text{MP}_k} = \frac{W}{R}$$

- The second equation is $y_0 = f(\ell^{\min}, k^{\min})$
- to get LRTC: $TC(W, R, y)$
- Example: find LRTC for $y = \ell^\alpha k^{1-\alpha}$ (CRS).

$$k = \frac{1-\alpha}{\alpha}\frac{W}{R}\ell$$

yielding *conditional* demand functions

$$\begin{aligned} \ell(W, R, y) &= \left(\frac{\alpha}{1-\alpha}\frac{R}{W}\right)^{1-\alpha} y \\ k(W, R, y) &= \left(\frac{1-a}{\alpha}\frac{W}{R}\right)^{\alpha} y \end{aligned}$$

yielding

$$\text{LRTC}(W, R, y) = W\ell + Rk = \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} R^{1-\alpha}W^\alpha + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} R^\alpha W^{1-\alpha}\right] y$$

Note: MC(y) =AC(y) is constant

3.4.3 Properties of Cost Functions

3.4.3.1 Relation between TC, AC, MC

As an example, consider the total cost function given by $TC(Q) = F + cQ^2$, $F, c \geq 0$. This cost function is illustrated on the left part of Figure 3.2. We refer to F as the *fixed cost* parameter,

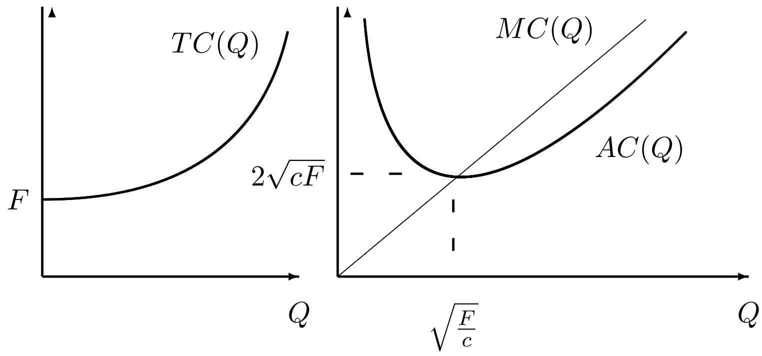

Figure 3.2: Total, average, and marginal cost functions

since the fixed cost is independent of the output level.

It is straightforward to calculate that $AC(Q) = F/Q + cQ$ and that $MC(Q) = 2cQ$. The average and marginal cost functions are drawn on the right part of Figure 3.2. The $MC(Q)$ curve is linear and rising with Q, and has a slope of $2c$. The $AC(Q)$ curve is falling with Q as long as the output level is sufficiently small ($Q < \sqrt{F/c}$), and is rising with Q for higher output levels ($Q > \sqrt{F/c}$). Thus, in this example the cost per unit of output reaches a minimum at an output level $Q = \sqrt{F/c}$.

We now demonstrate an "easy" method for finding the output level that minimizes the average cost.

Proposition 3.1 *If $Q^{\min} > 0$ minimizes $AC(Q)$, then $AC(Q^{\min}) = MC(Q^{\min})$.*

Proof. At the output level $Q^{\min}$, the slope of the $AC(Q)$ function must be zero. Hence,

$$0 = \frac{\partial AC(Q^{\min})}{\partial Q} = \frac{\partial \left(\frac{TC(Q^{\min})}{Q^{\min}}\right)}{\partial Q} = \frac{MC(Q^{\min})Q^{\min} - TC(Q^{\min})}{(Q^{\min})^2}.$$

Hence,

$$MC(Q^{\min}) = \frac{TC(Q^{\min})}{Q^{\min}} = AC(Q^{\min}).$$

■

We now return to our example illustrated in Figure 3.2, where $TC(Q) = F + cQ^2$. Proposition 3.1 states that in order to find the output level that minimizes the cost per unit, all that we need to do is extract $Q^{\min}$ from the equation $AC(Q^{\min}) = MC(Q^{\min})$. In our example,

$$AC(Q^{\min}) = \frac{F}{Q^{\min}} + cQ^{\min} = 2cQ^{\min} = MC(Q^{\min}).$$

Hence, $Q^{\min} = \sqrt{F/c}$, and $AC(Q^{\min}) = MC(Q^{\min}) = 2\sqrt{cF}$.

Do it in general (graphically only)

Another useful condition

$$\frac{W}{\text{MP}_\ell} = MC = \frac{R}{\text{MP}_k}$$

Proof.

$$\frac{dTC(y)}{d\ell} = \frac{dTC(f(\ell,k))}{d\ell} = \frac{\partial TC(y)}{\partial y}\frac{\partial y}{\partial \ell} = \text{MC}(y) \times \text{MP}_\ell$$

■

3.5 Profit Maximization

- Profit definition: $\pi = \text{TR} - \text{TC}$
- Two methods:
  1. choose the profit-maximizing output, y, using $TC(y)$
  2. choose the profit-maximizing factor employment using $f(\ell, k)$

3.5.1 Choosing profit-maximizing output

$$\max_y \pi(y) = TR(y) - TC(y) = p_y y - TC(y)$$

If $y^* > 0$, $p_y = \text{MC}(y^*)$
Condition needed: $p_y \geq ATC(y^*)$
Second order MC is declining with y.
Draw figures.

3.5.2 Choosing profit-maximizing factor employment

$$\pi = \text{TR} - \text{TC} = p_y f(\ell, k) - W\ell - Rk$$

yielding $W = p_y\text{MP}_\ell = \text{VMPL}$ and $R = \text{MP}_k = \text{VMPK}$
SOC:

$$f_{\ell\ell} < 0, \quad f_{kk} < 0, \quad \text{and } f_{\ell\ell}f_{kk} - (f_{\ell k})^2 > 0$$

3.5.3 Comparative statics: Equal-MP curves

TO BE COMPLETED