Intermediate Microeconomic Theory B
Undergraduate Lecture Notes

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Contents

Remarks

- Notes prepared during the 1st semester at the University of Haifa, March 2001 to June 2001 (Tash-Nach)
- For a Syllabus see a separate handout in Hebrew (summarized by the present Table of Content)
- Texts:
  1. Blumental, Levhari, Ofer, & Sheshinski. 1971. *Price Theory.* Academon Press.
  2. Varian H. 1987. *Intermediate Microeconomics.* W.W. Norton
  3. Shy, O. 1986. *Industrial Organization: Theory & Applications.* Cambridge, Mass.: The MIT Press
- Lecture is 3×45 minutes (given nonstop once a week

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Topic 1
Cost of Production

1.1 Cost Functions

- Let, W wage rate, R rental on capital
- $TC(W, R, y)$ maps factor-rental prices to \$s
- Emphasize duality: cost function can derived from a production function, and vise versa.
- Marginal Cost: $\mathrm{MC}(y) \stackrel{\text{def}}{=} \partial \mathrm{TC}(y)/\partial y$
- Average Cost: $\mathrm{AC}(y) \stackrel{\text{def}}{=} \mathrm{TC}(y)/y$

1.2 Single-factor case: A demonstration

How to derive the cost function from a production function $y = \ell^\gamma$? Let, W wage rate and $\gamma > 0$.

1. Input-requirement function: $\ell = y^{1/\gamma}$
2. cost means payment to factors: $TC(W, y) = W\ell = Wy^{1/\gamma}$.
3. Note: return to scale (see Figure 1.1):

$$(\lambda\ell)^\gamma > \lambda\ell^\gamma \quad \text{iff} \quad \gamma > 1$$

1.3 Cost minimization and long-run cost functions

- Given W and R, find ℓ and k that minimize cost of producing y_0 units of output.

$$\min_{\ell,k} W\ell + Rk \quad \textbf{s.t.} \quad f(\ell, k) \geq y_0$$

- Discuss corner vs. interior solutions. If $\ell^{\min}$, $k^{\min} > 0$,

$$\frac{\mathrm{MP}_\ell}{\mathrm{MP}_k} = \frac{W}{R}$$

- The second equation is $y_0 = f(\ell^{\min}, k^{\min})$
- to get LRTC: $TC(W, R, y)$

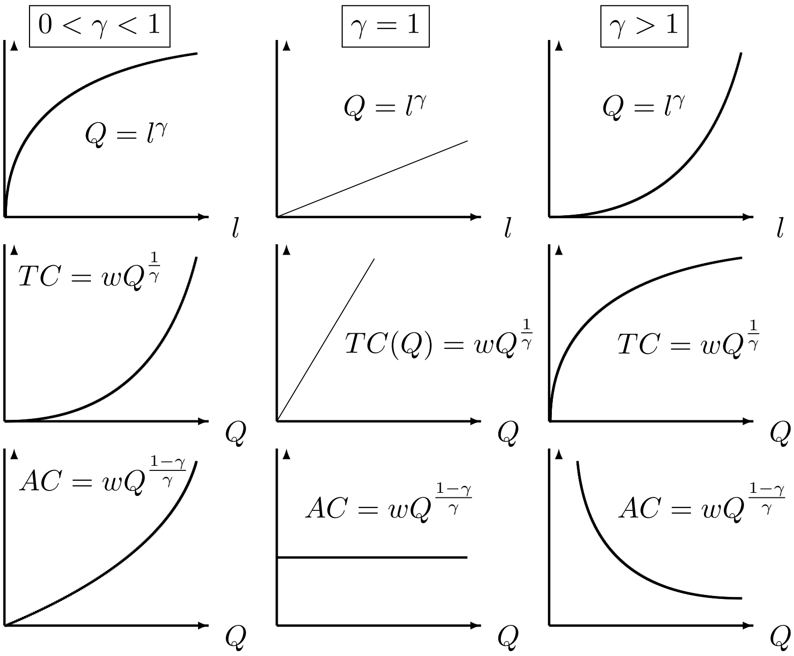

Figure 1.1: Duality between cost- and production functions

- Example: find LRTC for $y = \ell^\alpha k^{1-\alpha}$ (CRS).

$$k = \frac{1-\alpha}{\alpha}\frac{W}{R}\ell$$

yielding *conditional* demand functions

$$\begin{aligned}
\ell(W,R,y) &= \left(\frac{\alpha}{1-\alpha}\frac{R}{W}\right)^{1-\alpha} y \\
k(W,R,y) &= \left(\frac{1-a}{\alpha}\frac{W}{R}\right)^{\alpha} y
\end{aligned}$$

yielding

$$\text{LRTC}(W,R,y) = W\ell + Rk = \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} R^{1-\alpha}W^{\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} R^{\alpha}W^{1-\alpha}\right] y$$

Note: MC(y) =AC(y) is constant

1.4 Properties of Cost Functions

1.4.1 Relation between TC, AC, MC

As an example, consider the total cost function given by $TC(Q) = F + cQ^2$, $F, c \geq 0$. This cost function is illustrated on the left part of Figure 1.2. We refer to F as the *fixed cost* parameter,

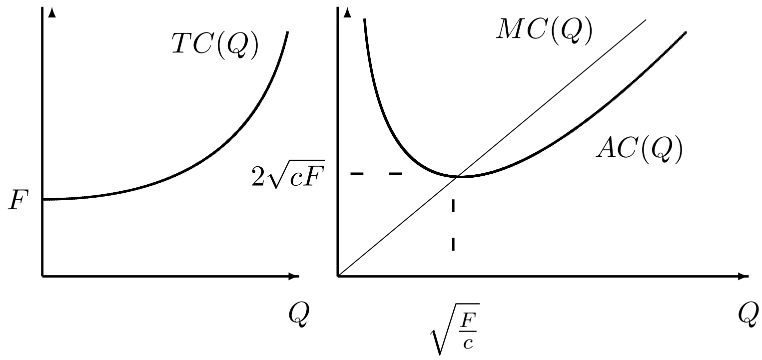

Figure 1.2: Total, average, and marginal cost functions

since the fixed cost is independent of the output level.

It is straightforward to calculate that $AC(Q) = F/Q + cQ$ and that $MC(Q) = 2cQ$. The average and marginal cost functions are drawn on the right part of Figure 1.2. The $MC(Q)$ curve is linear and rising with Q, and has a slope of $2c$. The $AC(Q)$ curve is falling with Q as long as the output level is sufficiently small ($Q < \sqrt{F/c}$), and is rising with Q for higher output levels ($Q > \sqrt{F/c}$). Thus, in this example the cost per unit of output reaches a minimum at an output level $Q = \sqrt{F/c}$.

We now demonstrate an "easy" method for finding the output level that minimizes the average cost.

Proposition 1.1 *If $Q^{\min} > 0$ minimizes $AC(Q)$, then $AC(Q^{\min}) = MC(Q^{\min})$.*

Proof. At the output level $Q^{\min}$, the slope of the $AC(Q)$ function must be zero. Hence,

$$0 = \frac{\partial AC(Q^{\min})}{\partial Q} = \frac{\partial\left(\frac{TC(Q^{\min})}{Q^{\min}}\right)}{\partial Q} = \frac{MC(Q^{\min})Q^{\min} - TC(Q^{\min})}{(Q^{\min})^2}.$$

Hence,

$$MC(Q^{\min}) = \frac{TC(Q^{\min})}{Q^{\min}} = AC(Q^{\min}).$$

■

We now return to our example illustrated in Figure 1.2, where $TC(Q) = F + cQ^2$. Proposition 1.1 states that in order to find the output level that minimizes the cost per unit, all that we need to do is extract $Q^{\min}$ from the equation $AC(Q^{\min}) = MC(Q^{\min})$. In our example,

$$AC(Q^{\min}) = \frac{F}{Q^{\min}} + cQ^{\min} = 2cQ^{\min} = MC(Q^{\min}).$$

Hence, $Q^{\min} = \sqrt{F/c}$, and $AC(Q^{\min}) = MC(Q^{\min}) = 2\sqrt{cF}$.

Do it in general (graphically only)

1.4.2 Another useful condition

$$\frac{W}{\mathrm{MP}_\ell} = MC = \frac{R}{\mathrm{MP}_k}$$

Proof.

$$\frac{dTC(y)}{d\ell} = \frac{dTC(f(\ell,k))}{d\ell} = \frac{\partial TC(y)}{\partial y}\frac{\partial y}{\partial \ell} = \mathrm{MC}(y) \times \mathrm{MP}_\ell$$

■

1.5 Optimal plant size

- Choosing the level of fixed costs (fixed factors)
- k denotes plant size (say k squared meters)
- $k(y)$ the optimal size plant given output level y
- Short run: $\mathrm{SRTC}(y,k)$
- $\mathrm{SRAC}(y,k) = \mathrm{SRTC}(y,k)/y$
- $\mathrm{LRTC}(y) = \mathrm{SRTC}(y,k(y))$
- Result: $\mathrm{LRTC}(y) \le \mathrm{SRTC}(y,k)$
- Plot envelope long run optimal plant AC, and SRACs for given values of k.

Topic 2
Profit Maximization

- Profit definition: $\pi = \text{TR} - \text{TC}$
- Two methods:
    1. choose the profit-maximizing output, y, using $TC(y)$
    2. choose the profit-maximizing factor employment using $f(\ell, k)$

2.1 Choosing profit-maximizing output

$$\max_y \pi(y) = TR(y) - TC(y) = p_y y - TC(y)$$

If $y^* > 0$, $p_y = \text{MC}(y^*)$
Condition needed: $p_y \geq ATC(y^*)$
Second order MC is declining with y.
Draw figures.

2.2 Choosing profit-maximizing factor employment

$$\pi = \text{TR} - \text{TC} = p_y f(\ell, k) - W\ell - Rk$$

yielding $W = p_y \text{MP}_\ell = \text{VMPL}$ and $R = \text{MP}_k = \text{VMPK}$
SOC:

$$f_{\ell\ell} < 0, \quad f_{kk} < 0, \quad \text{and } f_{\ell\ell} f_{kk} - (f_{\ell k})^2 > 0$$

Topic 3
Long-Run Supply and the Competitive Industry

3.1 Assumptions and Goals

- Each firm a competitive (price taker), thus faces a perfectly-elastic demand curve
- Long-run means free entry and exit
- Short-run, the number of firms is fixed (no entry or exit)
- Identical firms (not necessary, but will be assumed here)

The purpose of this analysis is

1. to calculate the long-run number of firms and aggregate output
2. to calculate the short run output level for a given number of firms

3.2 A Numerical Example

3.2.1 Data

- Industry faces the (inverse) demand curve: $p = 10000/Q$
- $n = 100$ identical firms: $\mathrm{TC}(q_i) = 50 + (q_i)^2/2$

3.2.2 Long-run equilibrium

$$\mathrm{MC} = q, \quad \mathrm{AC} = \frac{50}{q} + \frac{q}{2}.$$

$$\mathrm{MC} = \mathrm{AC} \Longrightarrow q_i = 10 \quad \text{and} \quad \min_q \mathrm{AC}(10) = 10.$$

Hence, the long-run industry supply is perfectly elastic at $p = 10$.

Intersecting demand and supply yields

$$p = \frac{10000}{Q} = 10 \Longrightarrow Q = 1000 \Longrightarrow q_i = \frac{Q}{n} = \frac{1000}{100} = 10.$$

3.2.3 The effect of a rise in fixed cost

Suppose that the government imposes a license fee of 15. We look for the new long-run equilibrium.

$$\mathrm{TC}(q_i) = 65 + \frac{(q_i)^2}{2}.$$

$$\mathrm{MC} = q, \quad \mathrm{AC} = \frac{65}{q} + \frac{q}{2}.$$

$$\mathrm{MC} = \mathrm{AC} \Longrightarrow q_i = \sqrt{130} \Longrightarrow \min_q \mathrm{AC}(q) = \frac{65}{130} + \frac{\sqrt{130}}{2} = \sqrt{130},$$

which is the industry's long-run supply curve.

$$p = \frac{10000}{Q} = \sqrt{130} \Longrightarrow Q = \frac{10000}{\sqrt{130}} \Longrightarrow n = \frac{Q}{q_i} = \frac{10000}{\sqrt{130}} = 76.92.$$

3.2.4 The effect of a rise in unit cost

Suppose that the government imposes a per-unit tax of 4.5. Calculate the *short-run* equilibrium (i.e., $n = 100$):

$$\mathrm{TC}(q_i) = 50 + \frac{(q_i)^2}{2} + 4.5q_i.$$

$$\mathrm{MC} = q + 4.5, \quad \mathrm{AC} = \frac{50}{q} + \frac{q}{2} + 4.5.$$

Now,

$$p = \frac{10000}{100q} = \frac{100}{q} = q + 4.5 = \mathrm{MC} \Longrightarrow q = 8 \Longrightarrow p = 12.5.$$

In the long run, the number of firms will decline.

$$q = 10 \Longrightarrow p = \mathrm{MC} = 10 + 4.5 = 14.5 = \frac{10000}{10n} \Longrightarrow n = \frac{1000}{14.5} = 68.96.$$

3.2.5 The effect of a demand shock

Suppose that the Ministry of Health declared the product to be unhealthy. Formally, the demand drops to $p = 6400/Q$. In the *short run*, $n = 100$. So, solve

$$p = \frac{6400}{100q} = q = \mathrm{MC} \Longrightarrow q = 8, \Longrightarrow Q = 800, \Longrightarrow p = \frac{6400}{800} = 8 < 10 = \min \mathrm{AC}.$$

Therefore, in the *long run*, the number of firms must decline.

$$p = q = 10 \Longrightarrow 10 = p = \frac{6400}{Q} \Longrightarrow Q = 640, \Longrightarrow n = \frac{Q}{q} = \frac{640}{10} = 64.$$

Topic 4
The Monopoly

4.1 Demand Characterization

- Faces the entire market demand curve
- Characterize TR(Q), MR(Q), for the inverse demand curve $p = a - bQ$
- Relate max TR to elasticity
- If the demand function is linear, $p = a - bQ$, then the marginal-revenue function is also linear, has the same intercept as the demand, but has twice the (negative) slope. Formally, $MR(Q) = a - 2bQ$.
- $$MR(Q) = p(Q)\left[1 + \frac{1}{\eta_p(Q)}\right].$$

  Proof.

  $$\begin{aligned} MR(Q) &\equiv \frac{\mathrm{d}\,TR(Q)}{\mathrm{d}Q} = \frac{\mathrm{d}[p(Q)Q]}{\mathrm{d}Q} = p + Q\frac{\mathrm{d}p(Q)}{\mathrm{d}Q} \\ &= p\left[1 + \frac{Q}{p}\frac{1}{\frac{\mathrm{d}Q(p)}{\mathrm{d}p}}\right] = p\left[1 + \frac{1}{\eta_p(Q)}\right]. \end{aligned}$$

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4.2 The Simple Monopoly

- Solve for the simply monopoly

$$\max_Q \pi = \mathrm{TR}(Q) - \mathrm{TC}(Q) \Longrightarrow \mathrm{MR}(Q) = \mathrm{MC}(Q) \quad \text{provided that} \quad p^m \geq \min \mathrm{MC}(Q^m).$$

Example: Figure 4.1 illustrates the monopoly solution for the case where $\mathrm{TC}(Q) = F + cQ^2$, and a linear demand function given by $p(Q) = a - bQ$. $\mathrm{MR}(Q) = a - 2bQ$. Hence, if $Q^m > 0$, then Q^m solves

$$\mathrm{MR}(Q) = a - 2bQ^m = 2cQ^m = \mathrm{MC}(Q)$$

implying that

$$Q^m = \frac{a}{2(b+c)} \quad \text{and hence} \quad p^m = a - bQ^m = \frac{a(b+2c)}{2(b+c)}.$$

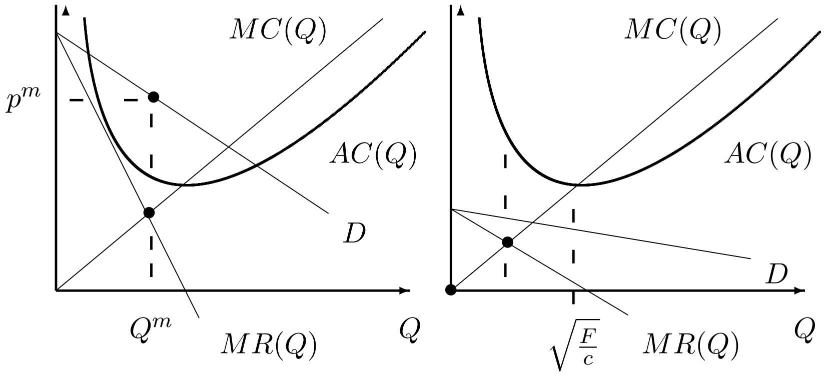

Figure 4.1: The monopoly's profit maximizing output

Consequently,

$$\begin{aligned}\pi(Q^m) &\equiv \text{TR}(Q^m) - \text{TC}(Q^m)\\ &= \frac{a^2(b+2c)}{4(b+c)^2} - F - c\left(\frac{a}{2(b+c)}\right)^2 = \frac{a^2}{4(b+c)} - F.\end{aligned}$$

Altogether, the monopoly's profit-maximizing output is given by

$$Q^m = \begin{cases} \dfrac{a}{2(b+c)} & \text{if } F \le \dfrac{a^2}{4(b+c)} \\ 0 & \text{otherwise .}\end{cases}$$

4.3 Discriminating Monopoly

- Selling to different markets (different demand curves)
- How to enforce anti-arbitrage measures (e.g., student discounts, senior citizens, hours of operation, late editions (book publishers))
- In some cases, it is profitable not to sell on some markets

Figure 4.2 illustrates the demand schedules in the two markets (market 1 and market 2).

- ΣMR is the horizontal sum of $\text{MR}_1 + \text{MR}_2$
- If it is profitable to serve *both* markets, then solution is found from $\Sigma\text{MR} = \text{MC}(q_1 + q_2)$
- Find q_1 and q_2 from $\text{MC}(q_1 + q_2) = \text{MR}_1(q_1) = \text{MR}_2(q_2)$
- Find p_1 and p_2 from each market demand curve
- It is NOT clear that it is profitable to serve market 1 (must be checked!)

Example: Two *segmented* markets: $q_1 = 2 - p_1$, and $q_2 = 4 - p_2$. Marginal cost is $c = 1$.

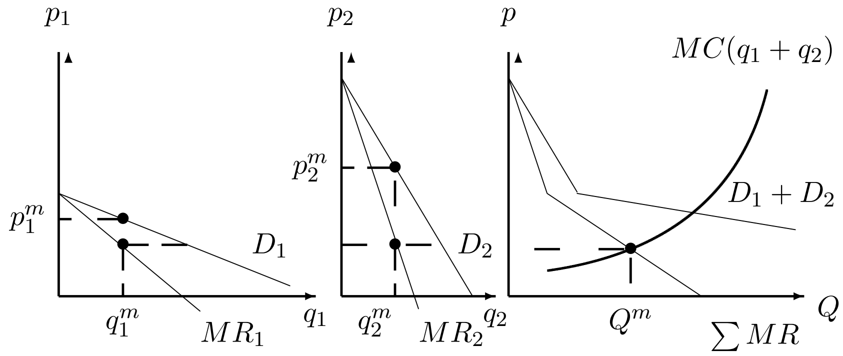

Figure 4.2: Monopoly discriminating between two markets

1. In market 1, $p_1 = 2 - q_1$. Hence, $\text{MR}_1(q_1) = 2 - 2q_1$. Equating $\text{MR}_1(q_1) = c = 1$ yields $q_1 = 0.5$. Hence, $p_1 = 1.5$.

   In market 2, $p_2 = 4 - q_2$. Hence, $\text{MR}_2(q_2) = 4 - 2q_2$. Equating $\text{MR}_2(q_2) = c = 1$ yields $q_2 = 1.5$. Hence, $p_2 = 2.5$.

2. $\pi_1 = (p_1 - c)q_1 = (0.5)^2 = 0.25$, and $\pi_2 = (p_2 - c)q_2 = (1.5)^2 = 2.25$. Summing up, the monopoly's profit under price discrimination is $\pi = 2.5$.

3. Suppose now that price discrimination is infeasible (markets are open). There are two cases to be considered: (i) The monopoly sets a uniform price $p \geq 2$ thereby selling only in market 2, or (ii) setting $p < 2$, thereby selling a strictly positive amount in both markets. Let us consider these two cases:

   (a) If $p \geq 2$, then $q_1 = 0$. Therefore, in this case the monopoly will set q_2 maximize its profit in market 2 only. By subquestion 1 above, $\pi = \pi_2 = 2.25$.

   (b) Here, if $p < 2$, $q_1 > 0$ and $q_2 > 0$. Therefore, aggregate demand is given by $Q(p) = q_1 + q_2 = 2 - p + 4 - p = 6 - 2p$, or $p(Q) = 3 - 0.5Q$. Hence, $MR(Q) = 3 - Q$. Equating $MR(Q) = c = 1$ yields $Q = 2$, hence, $p = 2$. Hence, in this case $\pi = (p - c)2 = 2 < 2.25$.

   Altogether, the monopoly will set a uniform price of $p = 2.5$ and will sell $Q = 1.5$ units in market 2 only.[1]

Finally, to find the relationship between the price charged in each market and the demand elasticities,

$$p_1^m(1 + 1/\eta_1) = p_2^m(1 + 1/\eta_2).$$

Hence, $p_2^m > p_1^m$ if $\eta_2 > \eta_1$, (or $|\eta_2| < |\eta_1|$, recalling that elasticity is a negative number). Hence, a discriminating monopoly selling a strictly positive amount in each market will charge a higher price at the market with the less elastic demand.

[1] Note that consumers in market 1 are better off under price discrimination than without it, since under no discrimination no output is purchased in market 1. Given that the price in market 2 is the same under price discrimination and without it, we can conclude that in this example, price discrimination is Pareto superior to nonprice discrimination, since both consumer surplus and the monopoly profit are higher under price discrimination.

4.4 The Cartel

- Contract among N competing firms
- Agreement on price or quantity quota (our focus)
- Examples: OPEC, IATA

The objective of the cartel is to choose $q_1, q_2, \ldots, q_N$ to

$$\begin{aligned} \max_{q_1, q_2, \ldots, q_N} \Pi(q_1, q_2, \ldots, q_N) &\equiv \sum_{i=1}^{N} \pi_i(q_i) \\ &= \left[a - b\sum_{i=1}^{N} q_i\right]\left(\sum_{i=1}^{N} q_i\right) - \sum_{i=1}^{N} TC_i(q_i). \end{aligned} \tag{4.1}$$

The cartel has to solve for N quantities, so, after some manipulations, the N first-order conditions are given by

$$0 = \frac{\partial \Pi}{\partial q_j} = a - 2b\sum_{i=1}^{N} q_i - MC_j(q_j) = MR(Q) - MC_j(q_j), \;\; j = 1, 2, \ldots, N. \tag{4.2}$$

4.4.0.1 A simple cartel example

- 10 firms, each has $\text{TC}(q_i) = 200 + 2(q_i)^2$
- Market demand: $p = 140 - Q$
- Solve for the cartel's output level, market price, and profit

$$\text{MR}(Q) = 140 - 2Q = 140 - 2 \cdot 10 \cdot q = 4q = \text{MC}(Q) \Longrightarrow q = \frac{35}{6}$$

Hence,

$$Q = 10 \cdot q = \frac{175}{3} \Longrightarrow p = 140 - Q = \frac{245}{3}.$$

Hence,

$$\pi_i = \left(\frac{245}{3}\right)\left(\frac{35}{6}\right) - 200 - 2\left(\frac{35}{6}\right)^2 = \frac{625}{3}. \Longrightarrow \Pi = 10\pi_i = \frac{6250}{3} \approx 208.$$

4.4.0.2 A more general example

- Our calculations will rely on $\text{TC}(q_i) = F + c(q_i)^2$
- Hence, $\text{MC}(q_i) = 2cq_i$ and $\text{AC}(q_i) = F/q_i + cq_i$
- Industry market demand: $p = a - bQ$

Since all plants have identical cost functions, we search for a symmetric equilibrium $q_1 = q_2 = \ldots = q_N \equiv q$. Hence,

$$a - 2bNq = 2cq \quad \text{implying that} \quad q = \frac{a}{2(bN + c)}. \tag{4.3}$$

The total cartel's output and the market price are given by

$$Q = Nq = \frac{Na}{2(bN + c)} \quad \text{and} \quad p = a - bQ = \frac{a(bN + 2c)}{2(bN + c)}. \tag{4.4}$$

4.5 Multiplant Monopoly

- Same as a cartel, but can adjust N (the number of producers/plants), since all under the same ownership.
- i.e., $\mathrm{MR}(Q) = \mathrm{MC}(q_i)$ for all $i = 1, \ldots, N$
- Choose q_i that minimizes $\mathrm{AC}(q_i)$

4.5.0.3 The simple multiplant-monopoly example

- Variable (controlled) number of firms, each has $\mathrm{TC}(q_i) = 200 + 2(q_i)^2$
- Market demand: $p = 140 - Q$
- Solve for # firms, output levels, market price, and profit
- Key issue: Here the monopoly adjusts output by changing the number of plants. In contrast, a cartel adjusts output by putting production quotas on member firms.

$$\mathrm{MC} = 4q_i, \quad \mathrm{AC} = \frac{200}{q_i} + 2q_i \Longrightarrow \arg\min \mathrm{AC}(q_i) = 10, \quad \min \mathrm{AC} = 40$$

Now,

$$\mathrm{MR} = 140 - 2Nq = 140 - 20N = 40 = \mathrm{MC} \Longrightarrow N = 5$$

Hence,

$$Q = 50, \Longrightarrow p = 90 \Longrightarrow \Pi = 90 \cdot 50 - 5 \cdot 400 = 2500$$

4.5.0.4 The more general example

- Hence, $q_i = \sqrt{F/c}$
- Solve $\mathrm{MR}(Q) = \mathrm{MC}(q_i)$
- Hence, $q_i = a/[2(bN + c)]$
- Altogether, $\sqrt{F/c} = a/[2(bN + c)]$, Hence,

$$N = \frac{a\sqrt{c}}{2b\sqrt{F}} - \frac{c}{b}$$

Topic 5
Oligopoly: Competition Among Few Firms

5.1 Noncooperative Game Theory: Nash Equilibrium

See Shy (1996), Chapter 2: pp.12–15, 18–20.

5.2 The Cournot Market Structure: Quantity Competition

5.2.1 Example of Cournot equilibrium

- Market demand: $Q = 3200 - 1600p$, Hence,

$$p = 2 - \frac{Q}{1600} = 2 - \frac{q_1 + q_2}{1600}$$

- 2 firms, firm 1 has a cost advantage:

$$\mathrm{TC}_1(q_1) = 0.25q_1 \qquad \mathrm{TC}_2(q_2) = 0.5q_2$$

- Solve for the Cournot output levels, market price, and profit levels

Firm 1 solves:

$$\max_{q_1} \pi_1 = \frac{3200 - q_1 - q_2}{1600} q_1 - 0.25q_1 \tag{5.1}$$

yielding a *best-response* function given by

$$q_1(q_2) = 1400 - \frac{1}{2}q_2$$

Firm 1 solves:

$$\max_{q_2} \pi_2 = \frac{3200 - q_1 - q_2}{1600} q_2 - 0.5q_2 \tag{5.2}$$

yielding a *best-response* function given by

$$q_2(q_1) = 1200 - \frac{1}{2}q_1 \tag{5.3}$$

Solving the two best-response functions yield

$$q_1 = \frac{3200}{3} \quad q_2 = \frac{2000}{3} \Longrightarrow Q = \frac{5200}{3} \approx 1733 \Longrightarrow p = \frac{11}{12}$$

Hence,

$$\pi_1 = \frac{6400}{9} \approx 711 > 278 \approx \frac{2500}{9} = \pi_2$$

5.2.2 General Cournot Theory

See Shy (1996), Chapter 6: pp.98–103.

5.3 Stackelberg Equilibrium: Sequential Moves

5.3.1 Example

- Two-stage game (two periods)
- Suppose now that firm 1 sets q_1 (stage I) before firm 2 sets q_2 (stage II)
- Firm 1 is called a *leader*
- Firm 2 is called a *follower* (choosing q_2 by taking q_1 as given)
- Solving the game *backwards* starting in the 2nd stage
- Stage II: Firm 2 takes q_1 as given and chooses q_2 to solve $\max_{q_2} \pi_2$ which is essentially the same as (5.2),
- yielding firm 2's best response function: (5.3)
- Stage I: Firm 1, knowing that firm 2 reacts according to (5.3)
- Thus, substitute (5.3) into (5.1), firm 1 solves

$$\max_{q_1} \pi_1 = \frac{3200 - q_1 - q_2(q_1)}{1600} q_1 - 0.25 q_1 = \frac{q_1(3200 - q_1)}{3200} \tag{5.4}$$

yielding

$$q_1 = 1600 > \frac{3200}{3} \Longrightarrow q_2 = 400 < \frac{2000}{3} \Longrightarrow Q = 2000 \Longrightarrow p = \frac{3200 - Q}{1600} = \frac{3}{4} < \frac{11}{12} \tag{5.5}$$

Also

$$\pi_1 = 800 > \frac{6400}{9} \qquad \pi_2 = 100 < \frac{2500}{9}$$

- Thus, the output and profit levels of firm 1 are higher than under Cournot
- The output and profit levels of firm 2 are lower than under Cournot
- Aggregate output is higher (hence, equilibrium price is lower)

5.3.2 General Stackelberg Theory

See Shy (1996), Chapter 6: pp.104–106

5.4 Bertrand Market Structure (price game)

- Each firm sets p_i (instead of q_i)
- Looking for (p_1, p_2) constituting a Nash equilibrium
- 2 firms, unit costs c_1 and c_2, where $c_1 \leq c_2$
- Solve for $c_1 = c_2 = c$ (identical firms)
- Solve for $c_1 < c_2$ with a small gap $c_2 - c_1$
- Solve for a large gap $c_2 - c_1$ (firm 1 charges a monopoly price)

5.5 Dominant Firm

- An industry having a single dominant firm and a fixed number of competitive firms in the short run (followers)
- Two-stage game: Stage I, leader the price p
- Stage II: Competitive firms take p as given and set competitive output level, q_i

Take the following example:

- One dominant firm with zero production cost
- 50 competitive firms, each with cost $\mathrm{TC}_i(q_i) = (q_i)^2/2$
- Market demand: $Q = 1000 - 50p$
- Calculate the price set by the dominant firm
- Calculate output levels of all firms

Second stage: Given p, find the competitive firms' aggregate supply curve:

$$p = \mathrm{MC}_i(q_i) = q_i \Longrightarrow q_i = p \Longrightarrow Q_c = 50p$$

First stage: The *residual* demand facing the dominant firm is:

$$q_d = Q - Q_c = 1000 - 50p - Q_c = 1000 - 50p - 50p = 1000 - 100p$$

The dominant firm, therefore, chooses p to solve

$$\max_p = pQ_d = p(1000 - 100p) \Longrightarrow p = 5$$

Hence,

$$q_d = 1000 - 500 = 500, \quad q_i = p = 5, \quad Q_c = 50 \cdot 5 = 250$$

Remark: in the long run, more firms will enter, and the leaders output will shrink.

Topic 6

Welfare Economics

- Define efficiency in the "weakest" possible sense (i.e., not to mix with a political definition)
- Characterize *market failure* situations where an outcome (equilibrium) is inefficient from a social view point
- Propose policy instruments (e.g., taxation) that will restore efficiency.

6.1 Pure Exchange Economy: Basic Definitions

- Pure exchange economy means no production (to be added later on)
- Prices are irrelevant for these definitions
- 2 persons: A and B
- 2 goods: X and Y
- x_A^0 initial *endowment* of good X to individual A. y_A^0, x_B^0, and y_B^0 are similarly defined
- Aggregate economy endowment (manna from heaven): $\bar{x} \stackrel{\text{def}}{=} x_A^0 + x_B^0$ and $\bar{y} \stackrel{\text{def}}{=} y_A^0 + y_B^0$
- x_A *allocation* of good X to individual A. x_B, y_A and y_B are similarly defined
- In class, draw Edgeworth Box

Definition 6.1 *An allocation is said to be **feasible** if $x_A + x_B = \bar{x}$ and $y_A + y_B = \bar{y}$.*

Definition 6.2 *A feasible allocation is said to be **Pareto Efficient** (Pareto Optimal) if there does not exist a different feasible allocation which makes at least one consumer better off and does not make any consumer worse off.*

Alternative definition:

Definition 6.3 *(a) A feasible allocation (x_A, y_A, x_B, y_B) is said to be **Pareto Superior** to allocation $(\hat{x}_A, \hat{y}_A, \hat{x}_B, \hat{y}_B)$ if*

$$U_A(x_A, y_A) \geq U_A(\hat{x}_A, \hat{y}_A) \tag{6.1}$$
$$U_B(x_B, y_B) \geq U_B(\hat{x}_B, \hat{y}_B) \tag{6.2}$$

where at least one strict inequality $>$ must hold.

*(b) A feasible allocation is said to be **Pareto Optimal** if there does not exist an allocation which is Pareto superior to it.*

6.2 Contract Curves

- Draw the **contract curve** for different preferences
- Prove that on any interior allocation on the contract curve,

$$\text{RCS}^A = \frac{\text{MU}_X^A}{\text{MU}_Y^A} = \frac{\text{MU}_X^B}{\text{MU}_Y^B} = \text{RCS}^B$$

- Example (interior contract curve): $U_A = x_A \cdot y_A$ endowed with $(3,2)$ and $U_B = x_B \cdot y_B$ endowed with $(1,6)$. Solution:

$$\frac{y_A}{x_A} = \frac{y_B}{x_B} = \frac{8 - y_A}{4 - x_A} \Longrightarrow y_A = 2x_A$$

- Example (non-interior contract curve, perfect substitutes): $U_A = x_A + 2y_A$, $U_B = 2x_B + 2y_B$, with $\bar{x} = 4$ and $\bar{y} = 8$. *Note:* Only the aggregate endowment matters for drawing the contract curve. Solution: 2 sides: $x_A = 0$ and $y_A + y_B = 8$; and $y_A = 8$ and $x_A + y_A = 4$.
- Example (perfect complements and Cobb-Douglas): $U_A = x_A \cdot y_A$ endowed with $(0, 10)$, and $U_B = \min\{x_B, y_B\}$ endowed with $(20, 5)$. Solution:

$$y_A = \begin{cases} 0 & \text{if } x_A \leq 5 \\ -5 + x_A & \text{if } 5 \leq x_A \leq 20 \end{cases}$$

- Example (perfect substitutes and Cobb-Douglas): $U_A = x_A + y_A$ endowed with $(60, 10)$; and $U_B = x_B \cdot y_B$ endowed with $(20, 30)$. Solution:

$$y_A = \begin{cases} 0 & \text{if } x_A \leq 40 \\ -40 + x_A & \text{if } 40 \leq x_A \leq 80 \end{cases}$$

- Difficult example (both consumes have perfect complements preferences): $U_A = \min\{x_A, y_A\}$ and $U_B = \min\{x_B, y_B\}$ with aggregate endowment of $\bar{x} = 20$ and $\bar{y} = 10$.
- As above but $U_A = \min\{2x_A, y_A\}$ and $U_B = \min\{x_B, y_B\}$
- As above buy $U_A = x_A + y_A$ and $U_B = x_B + y_B$. Solution: The contract curve is the entire box.

6.3 Efficient production

6.3.1 Basic conditions

- 2 factors (labor & capital), aggregate endowment $\bar{L}$ and $\bar{K}$
- 2 goods produced by $x = f(L_X, K_X)$ and $y = g(L_Y, K_Y)$
- Monotonicity implies full employment where $L_X + L_Y = \bar{L}$ and $K_X + K_Y = \bar{K}$

- Draw Edgeworth Box in the factor space and iso-quants
- Interior efficient production allocations (contract curve) satisfy

$$\mathrm{RTS}^X \stackrel{\text{def}}{=} \frac{\mathrm{MP}_L^X}{\mathrm{MP}_K^X} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_K^Y} \stackrel{\text{def}}{=} \mathrm{RTS}^Y$$

- If industry X contains more than one firm (say 2 firms) than efficient production allocation implies that

$$\mathrm{MP}_L^{X,\mathrm{I}} = \mathrm{MP}_L^{X,\mathrm{II}} \quad \text{and} \quad \mathrm{MP}_K^{X,\mathrm{I}} = \mathrm{MP}_K^{X,\mathrm{II}}$$

6.3.2 Capital and labor intensities in production

Definition 6.4 *The production of X is said to be* capital-intensive *relative to Y if*

$$\frac{K_X}{L_X} > \frac{K_Y}{L_Y}$$

- Draw contract curves reflecting unique and reversible intensities.
- Example: $X = (L_X)^{1/3}(K_X)^{2/3}$ and $Y = (L_Y)^{1/2}(K_Y)^{1/2}$

$$\frac{\mathrm{MP}_L^X}{\mathrm{MP}_K^X} = \frac{1}{2}\frac{K_X}{L_X} = \frac{K_Y}{L_Y} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_K^Y} \Longrightarrow \frac{K_X}{L_X} = 2\frac{K_Y}{L_Y} > \frac{K_Y}{L_Y}$$

  implying that the production of X is capital-intensive relative to Y

- To find the contract curve:

$$\frac{K_X}{L_X} = 2\frac{K_Y}{L_Y} > \frac{K_Y}{L_Y} \Longrightarrow K_X = \frac{2\bar{K}L_X}{\bar{L}+L_X} \xrightarrow[L_X \to \bar{L}]{} \bar{K}$$

6.3.3 Production efficiency and the production-possibility curve

- If the contract curve is a 1 : 1 function (i.e., not a set-valued function) then each allocation on the contract curve is associated with a unique point on the PPF
- Each allocation in the box which is *not* on the contract curve is associated with point inside the PPF
- Example: $x = \sqrt{L_X}\sqrt{K_X}$ and $y = \sqrt{L_Y}\sqrt{K_Y}$

$$\mathrm{RTS}^X \stackrel{\text{def}}{=} \frac{\mathrm{MP}_L^X}{\mathrm{MP}_K^X} = \frac{K_X}{L_X} = \frac{\bar{K}-K_X}{\bar{L}-L_X} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_K^Y} \stackrel{\text{def}}{=} \mathrm{RTS}^Y \Longrightarrow K_X = \frac{\bar{K}}{\bar{L}}L_X$$

  which is the contract curve. Now,

$$x = \sqrt{L_X}\sqrt{\frac{\bar{K}}{\bar{L}}L_X} = L_X\sqrt{\frac{\bar{K}}{\bar{L}}} \Longrightarrow L_X = x\sqrt{\frac{\bar{L}}{\bar{K}}}$$

$$y = \sqrt{\bar{L}-L_X}\sqrt{\bar{K}-\frac{\bar{K}}{\bar{L}}L_X} = (\bar{L}-L_X\sqrt{\frac{\bar{K}}{\bar{L}}}) \Longrightarrow L_X = \bar{L} - \sqrt{\frac{\bar{L}}{\bar{K}}}y$$

  Altogether,

$$y = \sqrt{\bar{K}\bar{L}} - x$$

- How to calculate the slope of the PPF: dy/dx?
- Claim: RPT_{XY} (rate of product transformation)

$$\mathrm{RPT}_{XY} \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_L^X} \quad \text{and} \quad \frac{\mathrm{MP}_K^Y}{\mathrm{MP}_K^X}$$

Proof.

$$\mathrm{MP}_L^X = \frac{\mathrm{d}x}{\mathrm{d}L} \quad \text{and} \quad \mathrm{MP}_L^Y = \frac{\mathrm{d}y}{\mathrm{d}L} \quad \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}L}}{\frac{\mathrm{d}x}{\mathrm{d}L}} = \frac{\mathrm{MP}_K^Y}{\mathrm{MP}_K^X}$$

6.4 Integrated economy (Full efficiency)

We look at interior conditions (although, we have already discussed cases in which allocations are not on the interior of Edgeworth Box.

consumption:

$$\mathrm{RCS}^A = \frac{\mathrm{MU}_X^A}{\mathrm{MU}_Y^A} = \frac{\mathrm{MU}_X^B}{\mathrm{MU}_Y^B} = \mathrm{RCS}^B$$

Production:

$$\mathrm{RTS}^X \stackrel{\mathrm{def}}{=} \frac{\mathrm{MP}_L^X}{\mathrm{MP}_K^X} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_K^Y} \stackrel{\mathrm{def}}{=} \mathrm{RTS}^Y$$

Industry:

$$\mathrm{MP}_L^{X,\mathrm{I}} = \mathrm{MP}_L^{X,\mathrm{II}} \quad \text{and} \quad \mathrm{MP}_K^{X,\mathrm{I}} = \mathrm{MP}_K^{X,\mathrm{II}}$$

Consumption-production: (representative indifference curve is tangent to the PPF)

$$\mathrm{RCS}_{XY} = \mathrm{RPT}_{XY}$$

Topic 7
Competitive Equilibrium and the Neoclassical Welfare Theorems

7.1 Competitive equilibrium

7.1.1 Definition

All agents: Consumers, producers, and factors of production are price takers. *Competitive equilibrium* is a price vector (p_X, p_Y, W, R) such that

(a) given these prices firms choose how much to produce, x^p and y^p to maximize profits

(b) given these prices firms choose hiring of factors of production, L_X, K_X, L_Y, K_Y, to maximize profits

(c) Consumers, given goods' prices, and their income from owning factors, choose how much to consumer by maximizing utility subject to their budget constrains, and choose (aggregate) consumption x^c, y^c

(d) Equilibrium in *all* markets: $x^c = x^p$, $y^c = y^p$, $L_X + L_Y = \bar{L}$, and $K_X + K_Y = \bar{K}$

7.1.2 Example: Pure exchange economy with 2 Cobb-Douglas

- Consumer A: $U_A = (x_A)^2 y_A$ endowed with $(2, 6)$
- Consumer B: $U_B = x_B y_B$ endowed with $(4, 2)$
- Solving for consumers' demand function yield

$$x_A = \frac{2}{3}\frac{I_A}{p_X} \quad y_A = \frac{1}{3}\frac{I_A}{p_Y} \quad x_B = \frac{1}{2}\frac{I_B}{p_X} \quad y_B = \frac{1}{2}\frac{I_B}{p_Y}$$

- Consumers' incomes are generated from selling their endowments:

$$I_A = 2p_X + 6p_Y \quad I_B = 4p_X + 2p_Y$$

- Market equilibrium conditions: $x_A + x_B = 2 + 4 = 6$ and $y_A + y_B = 6 + 2 = 8$
- Substituting incomes into equilibrium condition of market for X yield

$$\frac{2}{3}\left(2 + 6\frac{p_Y}{p_X}\right) + \frac{1}{2}\left(4 + 2\frac{p_Y}{p_X}\right) = 6$$

- Substituting incomes into the equilibrium condition of market Y yield

$$\frac{1}{3}\left(2\frac{p_X}{p_Y} + 6\right) + \frac{1}{2}\left(4\frac{p_X}{p_Y} + 2\right) = 8$$

- *Numeraire:* Since only the price ratio p_X/p_Y can be calculated, we can set $p_X \stackrel{\text{def}}{=} 1$ as a numeraire, so only p_Y remains to be solved!
- *Walras' Law:* Let there be n perfectly competitive markets. Then, if $n-1$ markets clear, the remaining market must also clear.
- In our case, $n = 2$, hence, if the market for X clears, the market for Y must clear.
- Proof for Walras Law: Implied by having all agents satisfying their budget constraints.
- solving the market equilibrium for Y yields

$$\frac{p_X}{p_Y} = \frac{15}{8} \Longrightarrow x_A = \frac{52}{15}, \quad x_B = \frac{38}{15}, \quad y_A = \frac{13}{4}, \quad y_B = \frac{19}{4}$$

7.2 The First-Welfare Theorem (the essence of capitalism)

7.2.1 The Theorem

Suppose that: (1) here are no externalities, (2) Complete markets: i.e., no missing markets Then, a competitive equilibrium allocation is Pareto efficient.
"Incomplete proof:"

Consumers:

$$\mathrm{RCS}^{\mathrm{A}} = \frac{\mathrm{MU}_X^A}{\mathrm{MU}_Y^A} = \frac{p_X}{p_Y} \quad \text{and} \quad \mathrm{RCS}^{\mathrm{B}} = \frac{\mathrm{MU}_X^B}{\mathrm{MU}_Y^B} = \frac{p_X}{p_Y}$$

Producers (input markets):

$$p_X \mathrm{MP}_L^X = W \quad p_X \mathrm{MP}_K^X = R \quad p_Y \mathrm{MP}_L^Y = W \quad p_Y \mathrm{MP}_K^Y = R$$

hence,

$$\mathrm{RTS}^{\mathrm{X}} = \frac{\mathrm{MP}_L^X}{\mathrm{MP}_K^X} = \frac{W}{R} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_K^Y} = \mathrm{RTS}^{\mathrm{Y}}$$

Producers & Consumers (output markets): Profit maximization implies:

$$\mathrm{RPT} = \frac{\mathrm{MP}_L^Y}{\mathrm{MP}_L^X} = \frac{\mathrm{MP}_K^Y}{\mathrm{MP}_K^X} = \frac{p_X}{p_Y} = \mathrm{RCS}$$

(draw the economy's iso-profit line tangent to the PPF, and to the indifference curve)

7.2.2 Why this theorem is so useful? an example of a single-person economy

- Goods: a product X, and leisure L (prices: p and w)
- Consumer: $U = x \cdot L$
- Firm: t = employment level. $x = \sqrt{t}$
- Production possibility curve: $x = \sqrt{24 - L}$

- Pareto Optimum implies

$$\mathrm{RPT} = -\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{24-L}} = \mathrm{RCS} = \frac{\mathrm{MU_L}}{\mathrm{MU_X}} = \frac{x}{L}$$

Hence,

$$\frac{1}{2\sqrt{24-L}} = \frac{x}{L} = \frac{\sqrt{24-L}}{L} \Longrightarrow L = 16, \quad x = \sqrt{8} = 2\sqrt{2}$$

which also a competitive equilibrium allocation

- Now, suppose that we did not have the First-Welfare Theorem!
- Profit maximization

$$\max_t \pi = px - wt = p\sqrt{t} - wt \Longrightarrow \sqrt{t^d} = \left(\frac{1}{2}\frac{p}{w}\right)$$

- To find the labor supply we need to calculate the firm's profit

$$\pi = p\sqrt{t} - wt = \frac{1}{4}\frac{p^2}{w}$$

- Consumer's income $I = 24w + \pi$
- Leisure demand (hence labor supply)

$$L = \frac{24w + \pi}{2w} \Longrightarrow t^s = 24 - L = \frac{24w - \pi}{2w} = 12 - \frac{1}{8}\frac{p^2}{w^2}$$

- Labor market equilibrium

$$t^s = 12 - \frac{1}{8}\frac{p^2}{w^2} = \left(\frac{1}{2}\frac{p}{w}\right) = t^d \Longrightarrow \frac{p}{w} = 2\sqrt{8} \Longrightarrow t = 8 \Longrightarrow x = \sqrt{8}$$

7.3 The Second-Welfare Theorem

- For every Pareto-optimal allocation there exist an initial endowment for which this Pareto-optimal allocation is a competitive equilibrium
- Demonstrate using an Edgeworth Box
- Explain the importance: Possibility of achieving any PO allocation via the competitive mechanism

7.4 Monopoly in Edgeworth Box

Demonstrate.

Topic 8
Externalities

8.1 Consumption Externalities

8.1.1 Definition

- General formulation: $U^A(x_A, y_A, x_B, y_B)$ and $U^B(x_A, y_A, x_B, y_B)$
- Define and explain the difference between *external economies* and *external diseconomies*
- A competitive equilibrium allocation need not be Pareto efficient. For example, $U^A(x_A, y_A)$ (selfish) but $U^B(x_B, y_B, x_A)$ (externality)
- Pareto optimality requires
$$\frac{\mathrm{MU}^A_{X_A}}{\mathrm{MU}^A_{Y_A}} = \frac{\mathrm{MU}^B_{X_B} - \mathrm{MU}^B_{X_A}}{\mathrm{MU}^B_{Y_B}}$$
Proof. Pareto optimal allocations are found solving
$$\max\ U^B(x_B, y_B, x_A) \quad \textbf{s.t.} \quad U^A(x_A, y_A) = U_0^A, \quad x_A + x_B = \bar{x}, \quad y_A + y_B = \bar{y}.$$
Define
$$L(\cdot) \overset{\text{def}}{=} U^B(\bar{x} - x_A, y_B, x_A) + \lambda\left[U_0^A - U^A(x_A, \bar{y} - y_B)\right].$$
The first-order conditions are given by
$$0 = \frac{\mathrm{d}L}{\mathrm{d}x_A} = -\mathrm{MU}^B_{x_B} + \mathrm{MU}^B_{x_A} - \lambda \mathrm{MU}^A_{x_A} \quad \text{and} \quad 0 = \frac{\mathrm{d}L}{\mathrm{d}y_B} = \mathrm{MU}^B_{y_B} + \lambda \mathrm{MU}^A_{y_A}$$
which yield the result.
- However, in a competitive equilibrium
$$\frac{\mathrm{MU}^A_{X_A}}{\mathrm{MU}^A_{Y_A}} = \frac{p_X}{p_Y} = \frac{\mathrm{MU}^B_{X_B}}{\mathrm{MU}^B_{Y_B}}$$

8.1.2 The Roomates' Example

- Two roommates: Tom and Jerry
- Two goods: Cookies and Music
- $U_T(c_T, m_T) = c_T + m_T$ and $U_J(c_J, m_T) = c_J - (m_T)^2/2$
- Endowment: each with 30 cookies, and 24 hours
- No communication allowed: $c_T = c_J = 30$ and $m_T = 24$ hours

- Pareto Optimality: $m_T = 1$. Why?

$$\max U_J = c_J - \frac{(m_T)^2}{2} \quad \textbf{s.t} \quad U_T^0 = c_T + m_T = 60 - c_J + m_T$$

Since $c_T = 60 - c_J$, hence $c_J = m_T$

$$\max_{m_T} \left(60 + m_T - U_T^0 - \frac{(m_T)^2}{2} \right) \Longrightarrow m_T = 1$$

- Suppose property rights belong to music lovers. Since $MRS^T = \mathrm{MU}_{c_T}^T / \mathrm{MU}_{m_T}^T = 1$, J can "bribe" T to reduce the playing time length from 24 to 1 by giving him 23 cookies.

8.1.3 Competitive Marriage

- Married couple: consumers A and B
- 3 goods: F = fish, E = Elvis music, and V = Vivaldi music
- Utility functions:

$$U_A = f_A + 120 \ln e_A - 60 v_B \quad \text{and} \quad U_B = f_B + 120 \ln v_B - 60 e_A$$

- Market prices: $p_f = 2$, $p_E = 6$, $p_v = 3$
- Incomes: $I_A = I_B = 100$
- Calculate a competitive equilibrium
- Calculate the Pareto-optimal allocations

8.2 Production Externalities

8.2.1 Definition

- Production externality occurs when the output level of one firm affects the production function (cost function) of a different firm.
- Examples: Pollution, infrastructure, training
- Distinguish from pecuniary (not real) externality which stems from equilibrium price effects

8.2.2 The Farmers' Example

Suppose that a honey farm is located next to an apple orchard, and each acts as a competitive firm. Let the amount of apples produced be measured by A and the amount of honey by H. The cost functions of the two firms are given by:

$$C_H(H) = \frac{H^2}{100}, \quad \text{and} \quad C_A(A) = \frac{A^2}{100} - H.$$

The fixed market price of honey is $p_H = \$2$ and the fixed market price of apples is $p_A = \$3$. Answer the following questions:

(a) Compute the equilibrium amount of honey and the number of apples produced under these prices. Calculate the profit of each firm and the sum of the firms' profits in this equilibrium.
Honey producer chooses H^e to $\max_H \pi_H = 2H - H^2/100$ yielding $H^e = 100$.
Apple producer takes $H^e = 100$ as given and chooses A^e to $\max_A \pi_A = 3A - A^2/100 + 100$ yielding $A^e = 150$. Hence, $\pi_H^e = 100$, $\pi_A^e = 325$, and $\pi_H^e + \pi_A^e = 425$.

(b) Suppose that the honey and apple firms merged. What would be the profit-maximizing amount of apples and honey produced. Calculate the profit of the merged firms. Will the merged firm increase or decrease the production of honey compared with the case where the honey producer is independent? Explain!
The merged firm chooses H^m and A^m to

$$\max_{\substack{H \geq 0 \\ A \geq 0}} \pi_H + \pi_A = 2H - \frac{H^2}{100} + 3A - \frac{A^2}{100} + H$$

Hence, $H^m = 150$ and $A^m = 150$. That is, due to the positive externality, the merged firm increases the production of honey (which reduces the cost of producing apples). Thus, $\pi_H + \pi_A = 450 > 425 = \pi_H^e + \pi_A^e$.

(c) Suppose now that merger is illegal, so each firm is forced to be independent. Is there any tax/subsidy on one of the producer that will bring the economy to produce the optimal amount of honey and apples?
Suppose that the government pays the honey producer a subsidy of s per-unit of honey produced. We now calculate the subsidy that induces the honey producer to produce $H = 150$. With the subsidy, the honey producer solves

$$\max_H \pi_H = 2H + sH - \frac{H^2}{100}, \quad \text{yielding} \quad s^* = \frac{H^*}{50} - 2 = \frac{150}{50} - 2 = 1.$$

8.2.3 The Airport Example

- Airport choosing the number of landings x
- Housing developer choosing # houses to build, y

$$\pi^A = 48x - x^2 \quad \text{and} \quad \pi^D = 60y - y^2 - xy$$

8.2.3.1 No regulations, no communication

Airport: $0 = 48 - 2x \Longrightarrow \; x = 24$.
Developer: $0 = 60 - 2y - x \Longrightarrow \; y = 18$.
Profit levels: $\pi^A = 576$; $\quad \pi^D = 324$; $\quad \Pi \stackrel{\text{def}}{=} \pi^A + \pi^D = 900$.

8.2.3.2 Pareto- (industry) optimal allocation

$$\max_{x,y} \Pi = 48x - x^2 + 60y - y^2 - xy$$

$$
\begin{aligned}
0 &= \frac{\partial \Pi}{\partial x} = 48 - 2x - y \\
0 &= \frac{\partial \Pi}{\partial y} = 60 - 2y - x
\end{aligned}
$$

Hence, $y^* = 24 \quad \text{and } x^* = 12$.

$$\pi^A = 432; \quad \pi^D = 576; \quad \Pi^* = 1008$$

8.2.3.3 "Strict prohibition:" developer has all property right (bargaining not allowed)

Hence, no planes, $x = 0$, $\pi^A = 0$.

$$\pi^D = 60y - y^2 - 0 \Longrightarrow \; 0 = 60 - 2y \Longrightarrow \; y = 30 \Longrightarrow \; \pi^D = 30^2 = 900 < \Pi^*.$$

Hence, this mechanism does not support PO.

8.2.3.4 Airport is liable for all damages (bad mechanism!)

Airport must pay xy to developer (compensation for the damage).

$$\max_x \pi^A = 48x - x^2 - xy = 48x - x^2 - 30x \Longrightarrow \; 0 = 18 - 2x \Longrightarrow \; x = 9$$

$$\pi^D = 60y - y^2 - xy + xy \Longrightarrow \; 0 = 60 - 2y \Longrightarrow \; y = 30$$

Hence,

$$\pi^A = 81; \quad \pi^D = 900; \quad \Pi = 981 < \Pi^*$$

This mechanism also does not support PO since the developer will over-produce to increase the compensation.

8.2.3.5 Coase equilibrium (bargaining after property rights are assigned to the airport): developer bribes the airport

The developer decides on x subject to leaving $\pi^A = 576$.
Level of the bribe $= 576 - (48x - x^2)$.

$$\max_{x,y} \pi^D = 60y - y^2 - xy - [576 - (48x - x^2)]$$

$$
\begin{aligned}
0 &= \frac{\partial \pi^D}{\partial y} = 60 - 2y - x \\
0 &= \frac{\partial \pi^D}{\partial x} = -y + 48 - 2x
\end{aligned}
$$

Hence, $y = 24 = y^*$ and $x = 12 = x^*$. *Pareto-allocation!!!*

Coase Theorem: if agents can bargain, then optimality is restored regardless of property rights assignment.

8.2.3.6 *Coase equilibrium (bargaining property rights are assigned to the Developer): Airport bribes the developer to allow him to increase x*

Airport has to leave the developer with at least $\pi^D \geq 900$ (see Subsection 8.2.3.4).
The compensation is $900 - (60y - y^2 - xy)$. Hence,

$$max_{x,y}\pi^A = 48x - x^2 - [900 - (60y - y^2 - xy)].$$

Yielding the same PO allocation with different profit levels.
Remark: Why property rights are needed? Answer: to define the reservation payoffs of the players before bargaining starts.

8.2.3.7 *Optimal tax*

- It is essential that the tax will be levied on the externality-producing activity directly!!!
- i.e., do not tax the food sold in airport restaurants, since consumers will bring their own food, but will not reduce travel. A tax on gas, will lead to refueling abroad.
- Exception: when the taxed item is a perfect complement to the externality activity.
- However, even if you tax the activity only, flights may switch to airports nearby in a different country.

Which tax on airport will restore optimality? i.e., $x^* = 12$, hence, $y^* = 24$.

$$\pi^A = 48x - x^2 - tx \Longrightarrow 0 = 48 - 2x - t^* \Longrightarrow t^* = 24$$

The rest follows.

8.2.4 Upstream-downstream Example

- Steel producer's cost function: $c_s(s, x)$, where s and x are quantity of steel and pollution produced
- Fisherman's cost function: $c_f(f, x)$
- A competitive steel factory solves

$$\max_{s,x} \pi_s = p_s \cdot s - c_s(s, x) \Longrightarrow p_s = \frac{\mathrm{d}c_s}{\mathrm{d}s} \quad \text{and} \quad 0 = \frac{\mathrm{d}c_s}{\mathrm{d}x}$$

- A competitive fisherman solves

$$\max_{f} \pi_f = p_f \cdot f - c_f(f, x) \Longrightarrow p_f = \frac{\mathrm{d}c_f}{\mathrm{d}f}$$

- Socially optimal outcome is found by maximizing joint profit

$$\max_{f,s,x} \Pi \stackrel{\text{def}}{=} \pi_s + \pi_f \Longrightarrow 0 = \frac{\mathrm{d}c_s}{\mathrm{d}x} + \frac{\mathrm{d}c_f}{\mathrm{d}x} \Longrightarrow \frac{\mathrm{d}c_s}{\mathrm{d}x} = -\frac{\mathrm{d}c_f}{\mathrm{d}x}$$

- Draw figure (see Varian)
- Numerical example (see Varian)

8.2.5 Advertising Externality

- In a shopping mall there are 2 stores. Advertising of one stores brings more customers to both stores
- a_1 advertising of store 1, a_2 advertising of store 2
- Profit functions: $\pi_1 = (60 + x_2)x_1 - 2(x_1)^2$ and $\pi_2 = (105 + x_1)x_2 - 2(x_2)^2$
- Solve for non-cooperative and collusion equilibria

Topic 9
Public Goods

9.1 Definition

- A good in which the consumption of one consumer does not exclude the consumption of other consumers.
- Examples: Broadcasting (radio & TV)
- Problematic examples: Bridges, roads (because of congestion)

9.2 Samuelson's Efficiency Condition

9.2.1 Efficiency

- Let X be public good and y be private good.
- Let $F(x,y)=0$ be the economy's production possibility curve
- Efficient allocation must satisfy

$$\mathrm{RCS}^A + \mathrm{RCS}^B = \frac{\mathrm{MU}_X^A}{\mathrm{MU}_{Y_A}^A} + \frac{\mathrm{MU}_X^B}{\mathrm{MU}_{Y_B}^B} = \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \mathrm{RPT}$$

- As opposed to having X private good where optimality requires that each consumer equates RCS to RPT.

9.2.2 Example: Transportation and Cobb-Douglas preferences

- 2 goods: $F =$ fish (private) and $T =$ train (transportation infrastructure)
- N people in town: $U_i \stackrel{\text{def}}{=} (f_i)^2 \cdot T$
- Production possibility frontier: $F^2 + 3T^2 = 1800$

Pareto-optimal provision of T is found from:

$$N \cdot \mathrm{MRS}^{\mathrm{i}} = N \cdot \frac{\mathrm{MU}_T^i}{\mathrm{MU}_{f_i}^i} = N \cdot \frac{f_i}{2T} = \mathrm{RPT} = 3\frac{T}{F} \Longrightarrow T = 30$$

9.2.3 Example: Fireworks

In Nahalal (a small town in Northern Israel) there are $N > 1$ people. Every year they have a fireworks show on the Israel's Independence Day. All consumers are identical, each consumer i consumers only 2 goods: a private good x_i and a public good F (fireworks). The utility of each consumer i is given by $u_i(x_i, F) = x_i + \sqrt{F}$. Let w_i denote the (fixed) income of consumer i. Suppose that the price of the private good is normalized to $p_x = \$1$, and that the cost of producing fireworks is $c(F) = F^2$.

(a) If fireworks is privately provided by a single competitive firm, calculate the total amount of fireworks they have on independence day. Does the amount of fireworks increase or decrease with the population size N? Explain!
Let p^f be the price of one firework. Suppose that consumer i buys f_i units of fireworks. Then, f_i solves

$$\max_{f_i} u_i = w_i - p^f f_i + \sqrt{f_i + \sum_{j \neq i} f_j}.$$

The first-order condition yields $p^f = 1/(2\sqrt{f_i + \sum_{j \neq i} f_j})$. The producer equates $p^f = MC(F) = 2F$. By symmetry, $f_i = f$ for all i, hence, $F = Nf$. Altogether,

$$F^e = \frac{1}{4^{\frac{2}{3}}} \quad \text{and} \quad f^e = \frac{1}{4^{\frac{2}{3}} N}$$

Thus, although the free-rider effect decreases the amount of fireworks purchased by each consumer when population increases, the aggregate level stays the same.

(b) What is the socially optimal amount of fireworks? Does it increase or decrease with N? Explain!
The social planner chooses a firework level F^* that solves the Samuelson condition given by

$$\sum_i \frac{1}{2\sqrt{F}} = N\frac{1}{2\sqrt{F}} = \mathrm{MC}(F) = 2F, \quad \text{hence, } F^* = \left(\frac{N}{4}\right)^{\frac{2}{3}} \text{ and } f^* = \frac{1}{4^{\frac{2}{3}} N^{\frac{1}{3}}}.$$

Here, the social planner increases the aggregate level when N increases to capture the benefit from an increased positive externality.

9.3 The Tragedy of the Commons

See a fishing example in Shy (1996), Ch.17, pp.448–451.