(1) Consider a system composed of two components labeled X and Y. There are two firms producing two different systems (different brands), at zero production cost. Firm A produces components X_A and Y_A, and firm B produces X_B and Y_B. In this market there are 100 consumers labeled AB, and 100 consumers labeled BA. The Utility function of a consumer i, j where $i, j = A, B$ is

$$U_{i,j} = \begin{cases} \beta - \left(p_i^X + p_j^Y\right) & \text{buys system } X_i Y_j \\ \beta - \left(p_j^X + p_j^Y\right) - \delta & \text{buys system } X_j Y_j \\ \beta - \left(p_i^X + p_i^Y\right) - \delta & \text{buys system } X_i Y_i \\ \beta - \left(p_j^X + p_i^Y\right) - 2\delta & \text{buys system } X_j Y_i \end{cases}$$

(1a) [8 pts.] Calculate the undercut-proof equilibrium prices assuming that the components produced by different firms are incompatible. *Hint:* First make sure that you know to define price-undercutting.

Incompatible systems implies that only systems $X_A Y_A$ and $X_B Y_B$ are available. Therefore, let $p_A \stackrel{\text{def}}{=} p_A^X + p_A^Y$ and $p_B \stackrel{\text{def}}{=} p_B^X + p_B^Y$.
Now, if $p_A = p_B$, consumers AB and BA are both indifferent between buying systems $X_A Y_A$ and $X_B Y_B$, because

$$U_{ij}(X_A Y_A) = \beta - p_A - \delta = \beta - p_B - \delta = U(ji) \quad \text{for } i, j = A, B, \quad i \neq j.$$

This means that firm A undercuts firm B when $p_A < p_B$,
and firm B undercuts firm B when $p_B < p_A$.
Altogether, in an UPE:

(a) firm A maximizes p_A subject to $\pi_B = 100 p_B \geq 200 p_A$, and

(b) firm B maximizes p_B subject to $\pi_A = 100 p_A \geq 200 p_B$.

The unique solution for the two equations (under =) are $p_A^I = p_B^I = 0$, where superscript "I" stands for incompatible systems.

(1b) [2 pts.] Calculate the equilibrium profit level of each firm.

$$\begin{aligned} \pi_A^I &= 100 p_A^I = 0 \\ \pi_B^I &= 100 p_B^I = 0 \end{aligned}$$

(1c) [8 pts.] Calculate the undercut-proof equilibrium prices assuming that the components produced by different firms are compatible.

Now the 4 systems: X_AY_A, X_BY_B, X_AY_B, and X_BY_A are available for purchase. We look for an equilibrium where type AB consumers buy system X_AY_B, where as type BA consumers buy system X_BY_A.

In the market for component X, UPE implies that

$$\begin{aligned} \pi_B^X = 100p_B^X &\geq 200(p_A^X - \delta) \\ \pi_A^X = 100p_A^X &\geq 200(p_B^X - \delta) \end{aligned}$$

yielding the unique X-component prices: $p_A^X = p_B^X = 2\delta$.

In the market for component Y, UPE implies that

$$\begin{aligned} \pi_B^Y = 100p_B^Y &\geq 200(p_A^Y - \delta) \\ \pi_A^Y = 100p_A^Y &\geq 200(p_B^Y - \delta) \end{aligned}$$

yielding the unique X-component prices: $p_A^Y = p_B^Y = 2\delta$.

(1d) [2 pts.] Calculate the equilibrium profit level of each firm.

$$\begin{aligned} \pi_A^C = \pi_A^X + \pi_A^Y &= 100 \cdot 2\delta + 100 \cdot 2\delta = 400\delta \\ \pi_B^C = \pi_B^X + \pi_B^Y &= 100 \cdot 2\delta + 100 \cdot 2\delta = 400\delta. \end{aligned}$$

(2) [10 pts.] You are given the following information about a market with two hardware brands labeled A and B:

(a) There are 100 A-oriented consumers, and 200 B-oriented consumers.

(b) Each consumer type has a utility function given by

$$U_A \stackrel{\text{def}}{=} \begin{cases} q_A - p_A & \text{buy } A \\ q_B - p_B - \delta & \text{buy } B, \end{cases} \quad U_B \stackrel{\text{def}}{=} \begin{cases} q_A - p_A - \delta & \text{buy } A \\ q_B - p_B & \text{buy } B, \end{cases}$$

where δ is the differentiation (switching cost) parameter, q_A is the number of A-buyers, and q_B is the number of B-buyers.

(c) Production is costless and in an undercut-proof equilibrium, brands' prices are

$$p_A = \frac{1485}{7} \quad \text{and} \quad p_B = \frac{1188}{7}.$$

Calculate the differentiation (switching-cost) parameter δ.

We know that in an UPE,

(a) firm A maximizes p_A subject to $\pi_B = 200 p_B \geq 300(p_A - \delta)$, and

(b) firm B maximizes p_B subject to $\pi_A = 100 p_A \geq 300(p_B - \delta)$.

Solving the first equation under $=$ yields

$$\delta = \frac{3p_A - 2p_B}{3} = 99.$$

Solving the second equation under $=$ yields

$$\delta = \frac{3p_B - p_A}{3} = 99.$$

There is an alternative solution taking into account network externalities. In this case, an UPE is the solution for

(a) firm A maximizes p_A subject to $\pi_B = 200 p_B \geq 300(p_A - \delta + 300 - 100)$, yielding $\delta = 299$, and

(b) firm B maximizes p_B subject to $\pi_A = 100 p_A \geq 300(p_B - \delta + 300 - 200)$ yielding $\delta = 199$.

(3) Consider a market for a popular software DOORS$^{\text{TM}}$. There are 100 support-oriented (type-O) consumers, and 100 support-independent (type-I) consumers, with utility functions given by

$$U^O \stackrel{\text{def}}{=} \begin{cases} 2q - p & \text{buys the software} \\ q & \text{pirates (steals) the software} \\ 0 & \text{does not use this software,} \end{cases} \quad \text{and } U^I \stackrel{\text{def}}{=} \begin{cases} q - p & \text{buys the software} \\ q & \text{pirates (steals) the software} \\ 0 & \text{does not use this software,} \end{cases}$$

where q denotes the number of users of this software (which includes the number of buyers and the number of pirates, if piracy takes place). Suppose that the software is costless to produce and costless to protect. Also, assume that DOORS$^{\text{TM}}$ provides support only to those consumers who buy the software.

(3a) [10 pts.] Suppose that DOORS$^{\text{TM}}$ is *not* protected, so piracy is an option for every consumer. Calculate the software seller's profit-maximizing price. Prove your answer.

Type I users pirate the software since $q - p < q$ for all $p > 0$. Hence, the number of users is at least 100.

Type O buy the software if $2(100 + 100) - p \geq 100 + 100$. Thus, to induce type O to buy the software, the price should not exceed $p = 200$.

The profit of the software publisher is $\pi^{\text{P}} = 100p = 20,000$.

(3b) [5 pts.] Suppose that DOORS$^{\text{TM}}$ is protected, so piracy is impossible. Calculate the software seller's profit-maximizing price. Prove your answer.

Piracy is no longer an option. If only type O buy the software, the price must satisfy $p \leq 2q = 200$.

At this price, type I users will also buy the software since $q - p = 100 + 100 - p \geq 0$.

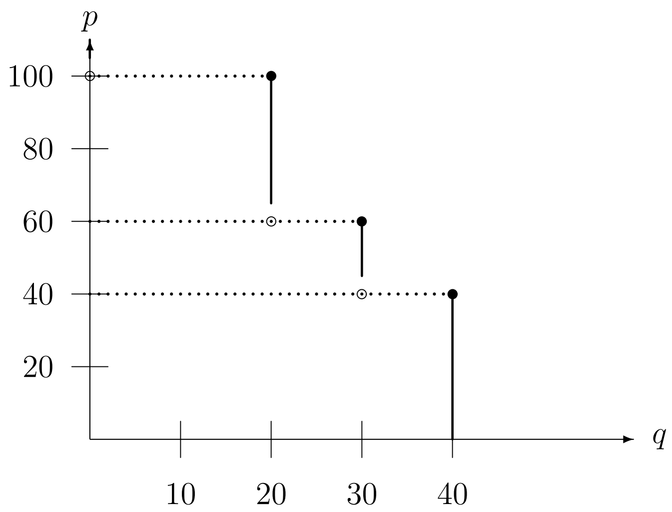
Hence, all users buy this software, so $q = 200$, hence $\pi^{\text{np}} = 200 \cdot 200 = 40,000 > 20,000$. Thus, DOORS$^{\text{TM}}$ should protect its software.

(4) Consider an economy with three types of consumers who wish to connect to a certain telecommunication service (e.g., obtaining a phone service). There are 20 type H consumers who place high value for connecting to this service, 10 type M consumers who place a lower value for this connection, and 10 type L consumers who place the lowest value on this service.

Let p denote the connection fee to this service, and q the actual number of consumers connecting to this service. Then, the utility function of each type

$$U_H \stackrel{\text{def}}{=} \begin{cases} 5q - p & \text{connected} \\ 0 & \text{disconnected} \end{cases} \text{ and } U_M \stackrel{\text{def}}{=} \begin{cases} 2q - p & \text{connected} \\ 0 & \text{disconnected.} \end{cases} \text{ and } U_L \stackrel{\text{def}}{=} \begin{cases} q - p & \text{connected} \\ 0 & \text{disconnected.} \end{cases}$$

(4a) [10 pts.] In the space below, draw the market demand function for connecting to this telecommunication service. Label the axes and prove and explain the graph.



(4b) [5 pts.] Suppose now that it costs the telephone company $\mu = 10$ to connect each consumer to this service. Calculate the connection price that maximizes the profit of this monopoly phone company.

$$\begin{aligned} p = 100 \Longrightarrow q = 20 \quad &\Longrightarrow \quad \pi = (100 - 10)20 = 1800 \\ p = 60 \Longrightarrow q = 30 \quad &\Longrightarrow \quad \pi = (60 - 10)30 = 1500 \\ p = 40 \Longrightarrow q = 40 \quad &\Longrightarrow \quad \pi = (40 - 10)40 = 1400. \end{aligned}$$

Clearly, the profit-maximizing connection fee is $p = 100$.

(5) Consider the broadcasted news scheduling model with three broadcasting stations labeled A, B, and C. There are η viewers whose ideal watching time is 17:00, and η viewers whose ideal watching time is 18:00. Let t_A denote the broadcasting time of station A, t_B the broadcasting time of station B, and t_C the broadcasting time of station C.
Assume that each station can air its news broadcast at one and only one time period. Also assume that each station earns exactly \$1 per viewer (as determined by rating surveys conducted during the broadcasting hours).

(5a) [5 pts.] List all the Nash equilibria in broadcasting time. (You do <u>not</u> have to provide a formal proof).

There are 6 Nash equilibria in broadcasting time given by

$$\begin{aligned}\langle t_A, t_B, t_C \rangle &= \langle 17, 17, 18 \rangle \\ &\quad \langle 17, 18, 17 \rangle \\ &\quad \langle 18, 17, 17 \rangle \\ &\quad \langle 18, 18, 17 \rangle \\ &\quad \langle 18, 17, 18 \rangle \\ &\quad \langle 17, 18, 18 \rangle.\end{aligned}$$

(5b) [5 pts.] Answer the previous question assuming that there are 3η viewers whose idea watching time is 17:00, and η viewers whose ideal watching time is 18:00.

The unique Nash equilibrium is $\langle t_A, t_B, t_C \rangle = \langle 17, 17, 17 \rangle$.

Proof.
In equilibrium, all viewers are equally distributed among the 3 stations, so

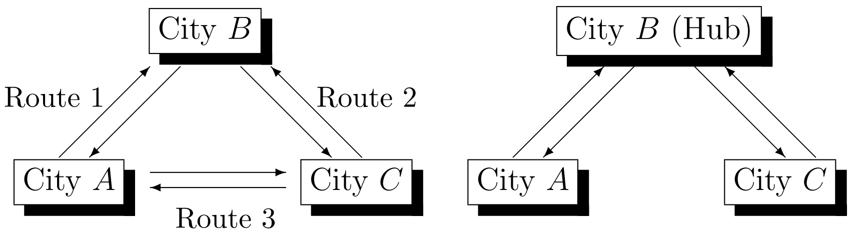
$$\pi_A = \pi_B = \pi_C = \frac{4\eta}{3}.$$

Suppose that station A deviates to $\tilde{t}_A = 18$. Then, only η viewers will watch station A, hence

$$\tilde{\pi}_A = \eta < \frac{4\eta}{3}$$

. Therefore, no station will unilaterally deviate to broadcast at $t = 18$.

(6) A single airline companies serves 3 cities as illustrated in the following figure.



On each route i, $i = 1, 2, 3$, there are η_i passengers. The cost of operating a flight on each route i is given by the function $c(q_i) = 1000 + \sqrt{q_i}$, where q_i is the actual number of passengers flying on route i. That is, each route requires a fixed cost of 1000 independent of the number of passengers plus variable cost which depends on the number of passengers q_i.

The airline considers two alternative networks of operations (displayed in the above figure): A fully-connected (FC), and a Hub-and-Spoke (with a hub in city B).

(6a) [5 pts.] Suppose $\eta_1 = \eta_2 = \eta_3 = \eta > 0$. Calculate which network of operation (FC or HS) minimized the airline's cost. Show your calculation!

Fully-connected network: $TC^{\text{FC}} = 3000 + 3\sqrt{\eta}$

Hub-and-spoke network: $TC^{\text{HS}} = 2000 + 2\sqrt{2\eta}$

Clearly, $TC^{\text{FC}} > TC^{\text{HS}}$ since even without taking fixed costs into account, $3\sqrt{\eta} > 2\sqrt{2\eta} \approx 2.82\sqrt{\eta}$.

(6b) [10 pts.] Suppose that the airline has already decided to operate a Hub-and-Spoke network. Also suppose that $\eta_1 = 100$ and $\eta_2 = \eta_3 = 800$. Does the airline minimize cost by locating the hub at city B? Prove your answer!

Denote by TC_{city} the airline's total cost when the hub is located in this city. Then,

$$\begin{aligned}
TC_A &= 2000 + \sqrt{100 + 800} + \sqrt{800 + 800} = 2000 + 30 + 40 = 2070 \\
TC_B &= 2000 + \sqrt{100 + 800} + \sqrt{800 + 800} = 2000 + 30 + 40 = 2070 \\
TC_C &= 2000 + \sqrt{100 + 800} + \sqrt{100 + 800} = 2000 + 30 + 30 = 2060.
\end{aligned}$$

Clearly, the hub should be located in city C.

(7) In an Island named *Bilingwa* off the coast of Mexico there are 100 inhabitants. 60 are native English speakers, whereas 40 are native Spanish speakers. Let n_{ES} denote the number of native English speakers who learn to speak Spanish. Similarly, let n_{SE} denote the number of native Spanish speakers who learn English. The utility of each residents increases with the number of residents to whom he is able to communicate with. We define the utility function of each native English and each native Spanish speakers, respectively, by

$$U_E = \begin{cases} \frac{60+n_{SE}}{10} & \text{does not learn Spanish} \\ \frac{60+40}{10} - 3 & \text{learns Spanish} \end{cases} \qquad U_S = \begin{cases} \frac{40+n_{ES}}{10} & \text{does not learn English} \\ \frac{40+60}{10} - 7 & \text{learns English} \end{cases}$$

These utility functions reveal that it is "easier" (less costly) for a native English speaker to learn Spanish, than for a native Spanish speaker to learn English (cost of 3 compared with 7).

(7a) [10 pts.] Find the number of native English speakers who learn Spanish, n_{ES}, and the number of native Spanish speakers who learn English, n_{SE} in a language-acquisition equilibrium. Is the equilibrium you found unique? Prove your results!

$\langle n_{ES}, n_{SE} \rangle = \langle 60, 0 \rangle$ is an equilibrium since:

$$(n_{SE} = 0) \Longrightarrow \frac{60+40}{10} - 3 > \frac{60}{10},$$

meaning that all E-speakers are better off learning S. In addition,

$$(n_{ES} = 60) \Longrightarrow \frac{40+60}{10} - 7 < \frac{40+60}{10},$$

meaning that given that all E-speakers learn S, there it is not beneficial for S-speakers to learn E. Finally, $\langle n_{ES}, n_{SE} \rangle = \langle 0, 40 \rangle$ is *not* an equilibrium since

$$(n_{ES} = 0) \Longrightarrow \frac{40+60}{10} - 7 < \frac{40}{10},$$

meaning that all S-speakers do not benefit from learning E even if E-speakers do not learn S.

(7b) [5 pts.] Find the socially-optimal levels of n_{ES} and n_{SE}. Prove your answer!

We define social welfare as the sum of utilities. Formally, let $W \stackrel{\text{def}}{=} 60U_A + 40U_B$. Then,

$$\begin{aligned} W(60,0) &= 60\left(\frac{100}{10} - 3\right) + 40\left(\frac{100}{10}\right) = 820 \\ W(0,40) &= 60\left(\frac{100}{10}\right) + 40\left(\frac{100}{10} - 7\right) = 720 \\ W(60,40) &= 60\left(\frac{100}{10} - 3\right) + 40\left(\frac{100}{10} - 7\right) = 540 \\ W(0,0) &= 60\left(\frac{60}{10}\right) + 40\left(\frac{40}{10}\right) = 520. \end{aligned}$$

Clearly, $\langle n_{ES}, n_{SE} \rangle = \langle 60, 0 \rangle$ is the socially optimal outcome.

THE END